Performance Comparison of Non-Binary LDPC Block and Spatially Coupled Codes

Lai Wei*, Kechao Huang†, David G. M. Mitchell*, Thomas E. Fuja*, and Daniel J. Costello, Jr.*
* Dept. of EE, University of Notre Dame, Notre Dame, IN, U.S, {lwei1, david.mitchell, tfuja, dcostel1}@nd.edu
† Dept. of ECE, Sun Yat-sen University, Guangzhou, China, hkech@mail2.sysu.edu.cn

I. Introduction

In this work, we consider two possible ways of improving the performance of binary low-density parity-check (LDPC) block codes. The first uses finite fields of size $q > 2$ to generate non-binary LDPC (NB-LDPC) block codes [1]. The performance gain introduced in this way comes at the cost of an increase in decoding complexity, and a variety of low-complexity algorithms have been proposed [2] [3] [4] to make NB-LDPC block codes practically feasible. The second modifies a binary LDPC block code by using an *edge spreading* technique [5] to introduce memory into the encoding process, thereby producing an LDPC convolutional code [6]. Terminated LDPC convolutional codes, which are also known as spatially-coupled LDPC (SC-LDPC) codes, have been shown to exhibit a phenomenon called "threshold saturation" [7] [8], which allows them to perform at the maximum *a posteriori* (MAP) threshold of the corresponding LDPC block codes, and thus to achieve capacity for increasing graph density, with belief propagation (BP) decoding.

As a result, a natural way to design good LDPC codes is to combine the two features discussed above to construct non-binary spatially-coupled LDPC (NB-SC-LDPC) codes. BP decoding thresholds of NB-SC-LDPC code ensembles on the binary erasure channel (BEC) were reported in [9] and [10], and threshold saturation was proved in [11]. In all these papers, flooding-schedule (FS) decoding was assumed, i.e., BP decoding was carried out across the entire parity-check matrix (graph) of the code simultaneously for a fixed number of iterations.

Here, we briefly summarize recent results obtained concerning (1) the BP threshold performance of NB-SC-LDPC code ensembles defined by *protographs* on the BEC and the binary-input additive white Gaussian noise channel (BIAWGNC), and (2) the bit-error-rate (BER) performance of finite-length NB-SC-LDPC codes based on protographs. In both cases, we assume that $q = 2^m$, where m is a positive integer, and we focus on the scenario when *windowed decoding* is used. Details of the results can be found in [12] and [13]. A brief introduction to key concepts and terminology now follows:

1) A protograph is a small Tanner graph that defines a structured LDPC code ensemble by placing constraints on both the degree distribution and the *edge connections* [14]. The threshold of a code ensemble is calculated directly based on the protograph, but the threshold calculation algorithms must be modified to take into account the particular edge connections.
2) To generate a finite-length NB-SC-LDPC code, the "copy-and-permute" or "graph lifting" procedure is applied to the protograph or base (parity-check) matrix representation of the protograph. Then, in the lifted graph or parity-check matrix, each edge or non-zero entry is randomly assigned a non-zero element from the non-binary finite field.
3) Windowed decoding (WD), proposed in [15], exploits the convolutional structure of SC-LDPC codes to "localize" the BP decoding process. By choosing a sufficiently large window size, the performance loss of WD can be made negligible, while large reductions in latency and memory requirements are realized compared to FS decoding.

This work was supported by the U.S. National Science Foundation under grant CCF-1161754.

II. Summary of Results: Threshold Analysis

We studied one $(2,4)$-regular NB-SC-LDPC code ensemble (referred to as $\mathcal{C}_1$) and two $(3,6)$-regular NB-SC-LDPC code ensembles (referred to as $\mathcal{C}_2$, with code memory, or coupling depth $m_s = 2$, and $\mathcal{C}_3$, with $m_s = 1$). $\mathcal{C}_1$ is generated based on the corresponding $(2,4)$-regular NB-LDPC block code ensemble $\mathcal{B}_1$ using the protograph representation and the edge-spreading technique, and $\mathcal{C}_2$ and $\mathcal{C}_3$ are generated based on the $(3,6)$-regular NB-LDPC block code ensemble $\mathcal{B}_2$ in a similar fashion. Assuming that the binary image of a non-binary codeword is transmitted, we focus on the threshold analysis of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ for both the BEC and the BIAWGNC when WD is used with different window sizes W and different finite field sizes q. Below, we denote the WD threshold of ensemble $\mathcal{C}_i$ $(i = 1, 2, 3)$ as $\delta_{\mathcal{C}_i}^{\mathrm{WD}}$ and the FS decoding threshold of ensemble $\mathcal{B}_j$ $(j = 1, 2)$ as $\delta_{\mathcal{B}_j}^{\mathrm{FS}}$. See [12] for details.

A. WD Thresholds on the BEC

We observed that for a sufficiently large W, $\delta_{\mathcal{C}_1}^{\mathrm{WD}}$ and $\delta_{\mathcal{C}_2}^{\mathrm{WD}}$ monotonically increase and saturate to a value numerically indistinguishable from channel capacity as q increases, while $\delta_{\mathcal{C}_3}^{\mathrm{WD}}$ monotonically increases to a near-capacity value as q increases, and then decreases slightly as q increases further; nevertheless, $\delta_{\mathcal{C}_3}^{\mathrm{WD}}$ stays very close to capacity even for very large q.

When W is sufficiently large, unlike the comparison of $\delta_{\mathcal{B}_1}^{\mathrm{FS}}$ and $\delta_{\mathcal{B}_2}^{\mathrm{FS}}$, $\delta_{\mathcal{C}_2}^{\mathrm{WD}}$ outperforms $\delta_{\mathcal{C}_1}^{\mathrm{WD}}$ for all q. Moreover, the gain of $\delta_{\mathcal{C}_1}^{\mathrm{WD}}$ compared to $\delta_{\mathcal{B}_1}^{\mathrm{FS}}$ and the gains of $\delta_{\mathcal{C}_2}^{\mathrm{WD}}$ and $\delta_{\mathcal{C}_3}^{\mathrm{WD}}$ compared to $\delta_{\mathcal{B}_2}^{\mathrm{FS}}$ all increase as q increases.

We define W^* as the window size such that, for any $W > W^*$, the WD thresholds remain almost identical for *all* q, i.e., little improvement is observed by choosing a window size beyond W^*. Clearly, the smaller W^* is, the more suitable an NB-SC-LDPC code ensemble is for WD. Numerical results show that, among $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, ensemble $\mathcal{C}_3$ has the smallest W^*. We conclude that $\mathcal{C}_3$ (with $m_s = 1$) is the best choice for WD due to its near-capacity threshold performance, even when both W and q are relatively small.

B. WD Thresholds on the BIAWGNC

The WD threshold performance of the same NB-SC-LDPC code ensemble examples on the BIAWGNC is consistent with the observations made for the BEC.

C. Decoding Complexity

When designing finite-length NB-SC-LDPC codes, we are interested in their performance compared to the corresponding NB-LDPC block codes when the decoding latency is the same, i.e., the window size in WD of NB-SC-LDPC codes and the blocklength of NB-LDPC block codes (both measured in *bits*) should be equal. Under this constraint, it is reasonable to estimate the complexity of NB-SC-LDPC codes *per window*. Numerical results further strengthen the conclusions in Sections II-A and II-B: the $(3, 6)$-regular construction of NB-SC-LDPC codes (i.e., from ensembles $\mathcal{C}_2$ and $\mathcal{C}_3$, especially $\mathcal{C}_3$) is better than the $(2, 4)$-regular construction (i.e., from ensemble $\mathcal{C}_1$) with decoding latency and complexity constraints and near-capacity performance (note that $\mathcal{B}_1$ and $\mathcal{B}_2$ cannot provide near-capacity threshold performance).

III. Summary of Results: Performance of Finite-length Codes

We also compared the finite-length performance of NB-SC-LDPC codes with NB-LDPC block codes on the BIAWGNC from two points of view. One is when the constraint length of the NB-SC-LDPC codes is equal to the blocklength of the NB-LDPC block codes, and the other is when the decoding latency of the NB-SC-LDPC codes is equal to the blocklength of the NB-LDPC block codes. For a fair comparison, the parity-check matrix of the SC-LDPC codes was derived by unwrapping [6] the corresponding parity-check matrix of the LDPC block codes. See [13] for details.

A. Convolutional Gain

We observed that the $(2, 4)$-regular NB-SC-LDPC codes with short constraint lengths (corresponding to small protograph lifting factors M) achieve significant convolutional gains compared to the underlying LDPC block codes, where the gains decrease as M increases. Further, the $(3, 6)$-regular NB-SC-LDPC codes achieve even larger convolutional gains compared to the underlying LDPC block codes, where the gains again decrease as M increases. These results are consistent with the threshold performance analysis in Section II.

B. Equal-Latency Comparison

The WD threshold analysis shows that the $(3, 6)$-regular NB-SC-LDPC code ensemble $\mathcal{C}_3$ (with $m_s = 1$) has near-capacity performance, even when both q and W are small. Thus, we focus on the comparison of the $(3, 6)$-regular NB-SC-LDPC codes in the ensemble $\mathcal{C}_3$ to the corresponding NB-LDPC block codes on the basis of equal decoding latency. Given a decoding latency, increasing W results in a smaller M; note that the decoder performance improves as W increases, while the code performance improves as M increases. Under the equal-latency constraint, we observed that $(3, 6)$-regular NB-SC-LDPC codes outperform both binary and non-binary LDPC block codes and binary SC-LDPC codes. Moreover, for a fixed q, the performance of $(3, 6)$-regular NB-SC-LDPC codes improves as W increases, and then worsens slightly as W increases further. For a fixed W, the performance improves as q increases.

References

[1] M. C. Davey and D. J. C. MacKay, "Low-density parity check codes over $GF(q)$," *IEEE Commun. Letters,* vol. 2, no. 6, pp. 165-167, June 1998.

[2] L. Barnault and D. Declercq, "Fast decoding algorithm for LDPC over GF(2^q)," in *Proc. IEEE Inf. Theory Workshop*, pp. 70-73, Paris, France, Apr. 2003.

[3] A. Voicila, D. Declercq, F. Verdier, M. Fossorier, M., and P. Urard, "Low-complexity decoding for non-binary LDPC codes in high order fields," *IEEE Trans. Commun.,* vol. 58, no. 5, pp. 1365-1375, May 2010.

[4] Erbao Li, D. Declercq, and K. Gunnam, "Trellis-based extended min-sum algorithm for non-binary LDPC codes and its hardware structure," *IEEE Trans. Commun.,* vol. 61, no. 7, pp. 2600-2611, July 2013.

[5] M. Lentmaier, G. P. Fettweis, K. Sh. Zigangirov, and D. J. Costello, Jr., "Approaching capacity with asymptotically regular LDPC codes," in *Proc. Inf. Theory and App. Workshop,* San Diego, CA, Feb. 2009.

[6] A. J. Felström and K. Sh. Zigangirov, "Time-varying periodic convolutional codes with low-density parity-check matrix," *IEEE Trans. Inf. Theory*, vol. 45, no. 6, pp. 2181–2191, Sept. 1999.

[7] M. Lentmaier, A. Sridharan, D. J. Costello, Jr., and K. Sh. Zigangirov, "Iterative decoding threshold analysis for LDPC convolutional codes," *IEEE Trans. Inf. Theory*, vol 56, no. 10, pp. 5274-5289, Oct. 2010.

[8] S. Kudekar, T. J. Richardson, and R. L. Urbanke, "Threshold saturation via spatial coupling: Why convolutional LDPC ensembles perform so well over the BEC," *IEEE Trans. Inf. Theory,* vol. 57, no. 2, pp. 803-834, Feb. 2011.

[9] H. Uchikawa, K. Kasai, and K. Sakaniwa, "Design and performance of rate-compatible non-binary LDPC convolutional codes," 2011. [Online]. Available: http://arxiv.org/pdf/1010.0060v2.pdf

[10] A. Piemontese, A. G. Amat, and G. Colavolpe, "Nonbinary spatially-coupled LDPC codes on the binary erasure channel," in *Proc. IEEE Int. Conf. Commun.,* pp. 3270-3274, Budapest, Hungary, June 2013.

[11] I. Andriyanova and A. G. Amat, "Threshold saturation for nonbinary SC-LDPC codes on the binary erasure channel," 2013. [Online]. Available: http://arxiv.org/abs/1311.2003/

[12] L. Wei, T. K. Akino, D. G. M. Mitchell, T. E. Fuja, and D. J. Costello, Jr., "Threshold analysis of non-binary spatially coupled codes with windowed decoding," submitted to *IEEE Int. Symp. on Inf. Theory*, Jan. 2014.

[13] K. Huang, D. G. M. Mitchell, L. Wei, X. Ma, and D. J. Costello, Jr., "Performance comparison of non-binary LDPC block and spatially coupled codes," submitted to *IEEE Int. Symp. on Inf. Theory,* Jan. 2014.

[14] J. Thorpe, "Low-density parity-check (LDPC) codes constructed from protographs," JPL IPN Progress Report 42-154, Aug. 2003.

[15] A. R. Iyengar, M. Papaleo, P. H. Siegel, J. K. Wolf, A. Vanelli-Coralli, and G. E. Corazza, "Windowed decoding of protograph-based LDPC convolutional codes over erasure channels," *IEEE Trans. Inf. Theory,* vol. 58, no. 4, pp. 2303-2320, Apr. 2012.