Compressive Interferometry for Optical Modal Analysis in Arbitrary Degrees of Freedom

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Abstract—Modal analysis of an optical beam refers to analyzing the beam into its constituent modes. In this paper, a new compressive approach for modal analysis of sparse beams using generalized interferometry in arbitrary degrees of freedom (DoFs) is proposed. It is shown that the modal content can be recovered by solving a sparse recovery problem based on random interferometric measurements obtained by sweeping the parameter(s) of unitary optical transformations that generalize the traditional temporal delay to an arbitrary DoF. The proposed compressive approach holds significant potential to reduce the computational complexity for modal analysis of optical beams in arbitrary DoFs using native optics hardware, namely two-path interferometry, with no need for extra hardware.

Index Terms—Modal Analysis, Compressed Sensing, Interferogram, Sparse Recovery.

I. INTRODUCTION

A beam of light generally consists of a combination of optical modes related to different Degrees of Freedom (DoFs), such as time, space and polarization. For example, an optical pulse consists of the electromagnetic field of a linear combination of different frequency harmonics in the temporal DoF.

Modal analysis refers to the process whereby a light beam may be analyzed into its constituent modes [1]. Analyzing the modal content of an optical beam has immediate bearing on a host of applications. Examples include optical fiber communications wherein the light at the receiving end may consist of several linearly polarized fields with different indices [2]. Also, in free-space optical communications, the transmitted light could be obtained by multiplexing several modes with different Orbital Angular Momentum (OAM) [3]. In such applications, achieving high-rate communications exploiting multimodal diversity relies on our ability to analyze a received beam into its constituent modes.

Optical interferometry is an important approach to investigate the nature and properties of a coherent beam of light. In temporal interferometry, a pulse of light enters a Mach-Zehnder interferometer (MZI) [2] to produce an interferogram by interfering the pulse with a time-delayed version of itself. The Fourier Transform (FT) of this interferogram can reveal the spectral content of the light beam, i.e., the modal coefficients in the temporal DoF. Recently, Abouraddy et al. devised a new approach, called generalized interferometry whereby beams can be analyzed in arbitrary DoFs [4], [1]. This is achieved by replacing the delay in MZI with an appropriate unitary optical transformation known as Generalized phase operator (GPO). This “generalized delay” is a continuous real parameter. Hereafter, we use the term “delay” to refer to generalized delay in arbitrary DoF.

While modal analysis based on interferograms is a novel and simple approach, there are some difficulties associated with it. In particular, accessing a full interferogram is computationally prohibitive. In practice, measurements are acquired by varying the delay and recording successive intensity measurements. Generally, such samples will have to be acquired at the Nyquist rate to avoid aliasing in modal analysis. On one hand, this requires the collection of a large number of samples, implying more latency, which may be intolerable for delay-sensitive applications that require real time processing such as with characterization of spatial scene dynamics. On the other hand, accuracy is compromised when stringent requirements on the sampling rates cannot be met due to practical hardware constraints.

In this paper, we devise a new compressive approach for modal analysis of optical beams in arbitrary DoF from compressive interferograms obtained using the generalized interferometry mechanism developed in [4], [1]. This approach leverages the sparse representation of optical beams in the frequency domain by posing the modal analysis problem as a sparse recovery problem. Since only few modes are present, the interferogram has a sparse representation in the frequency domain with non-zero frequency coefficients corresponding to the active modes. In contrast to the traditional approach wherein a fixed sampling period is used, we sample the interferogram by randomly choosing phase shifts. The collected measurements consist of linear combinations of the modal energies. Equivalently, the acquired measurements are obtained by applying a random matrix transformation to the sparse vector of modal energies. While this measurement matrix only has few degrees of freedom due to the randomly chosen phase shifts, it is shown that the modal energies can be efficiently recovered using standard sparse reconstruction algorithms such as Basis Pursuit (BP) [5], Matching Pursuit (MP) [6], or Smoothed ℓ0-norm (SLO) [7].

It is important to emphasize that recent work has made use of compressive sensing (CS) approaches for optical field estimation and imaging - and not for modal analysis - by introducing a sequence of random masks in the field path [8], [9], [10]. In contrast, we exploit CS here in the context of native optics hardware, namely two-path interferometry, for modal analysis with no need for extra hardware. The original interferometry setup with an appropriate optical transformation

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is used to obtain interferometric measurements, which are processed to reconstruct the modal energies via CS recovery algorithms.

The rest of the paper is organized as follows. In Section II, we state the modal analysis problem based on generalized interferometry. Modal analysis is posed as a sparse recovery problem and a compressive approach is proposed in Section III. In Section IV, we provide numerical results for three different examples including 1D and 2D cases. We conclude in Section V.

II. PROBLEM STATEMENT

In this section, we describe the modal composition of a light beam and the interferogram model for the 1-D and 2-D cases. In the 1-D case, the field can be written as an linear combination of the elements of an arbitrary complete orthonormal basis as

$$E(x) = \sum_{n=1}^{N} c_n \psi_n(x),$$

(1)

where $\{\psi_n(x)\}_n$ is an orthonormal basis corresponding to the DoF $x$ and $c_n, n = 1, ..., N$, are the modal coefficients. The non-zero coefficients correspond to the existing modes. Examples of such a basis include the time and spatial harmonics, i.e., $\{e^{-j\omega t}\}_\omega$ or $\{e^{-jkx}\}_k$, respectively. Generally speaking, the orthonormal basis related to any DoF could be either a countable or an uncountable set of modes.

In traditional temporal interferometry, a beam splitter directs two copies of the same beam to different arms of an interferometer (e.g. MZI). One arm introduces some delay $\alpha$ so that the beam can interfere with a time-delayed version of itself. In generalized interferometry [1], this idea is extended to arbitrary DoFs, namely, one arm is equipped with an appropriate GPO,

$$\Lambda(x,x';\alpha) = \sum_{n=1}^{N} e^{jn\alpha} \psi_n(x) \psi_n^*(x'),$$

(2)

which introduces a phase shift (delay) $\alpha$ related to the modes indexed by $n$. Hence, the output field of the GPO can be written as

$$E_\alpha(x;\alpha) = \int \Lambda(x,x';\alpha) E(x') dx' = \sum_{n=1}^{N} e^{jn\alpha} c_n \psi_n(x).$$

(3)

For the given $\alpha$, interfering the output fields of both arms results in an interferogram

$$P(\alpha) = 1 + \Re \int E(x) E_\alpha^*(x;\alpha) dx = 1 + \sum_{n=1}^{N} |c_n|^2 \cos(n\alpha).$$

(4)

Thus, it is straightforward to recover the modal coefficients by taking a Fourier transform (FT) of $P(\alpha)$.

In the case of two modal bases (corresponding to two different DoFs), the field is given by

$$E(x,y) = \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} c_{nm} \psi_n(x) \phi_m(y),$$

(5)

where, $\{\psi_n(x)\}_n$ and $\{\phi_m(y)\}_m$ are bases corresponding to DoF $x$ and $y$, respectively, and $c_{nm}, n = 1, ..., N_1; m = 1, ..., N_2$, are the modal coefficients. Orbital Angular Momentum (OAM) modes corresponding to states of $l \hbar$ OAM per photon and Laguerre Gaussian (LG) modes are two other examples of optical bases in angular and radial DoFs $\theta$ and $\rho$, respectively. Extending generalized interferometry to the 2-D case, two different GPOs are employed to introduce delays in the different DoFs. Thus, the interferogram for the 2-D case is given by

$$P(\alpha,\beta) = 1 + \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} |c_{nm}|^2 \cos(n\alpha - m\beta),$$

(6)

when the GPOs are placed in different arms, or

$$P(\alpha,\beta) = 1 + \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} |c_{nm}|^2 \cos(n\alpha + m\beta),$$

(7)

when the GPOs are in the same arm. Also, $\alpha$ and $\beta$ are the phase shifts corresponding to the DoFs (see [4] and [1] for more details).

Based on (4), (6), and (7), we see that the squared magnitudes of the modal coefficients can be recovered using the FT of the interferogram. However, this is typically associated with a high computational complexity in addition to practical difficulties, since a large number of interferogram samples need to be acquired at the Nyquist rate in order to efficiently recover the modal content. To address these problems, we propose a compressive interferometry approach that leverages the sparsity of optical beams.

III. COMPRESSION INTERFEROMETRY

Optical beams often consist of a small number of modes, wherefore most of the modal coefficients are zero. In such case, the light beam admits a sparse representation in the space defined by these modal components. This sparsity property enables us to model the modal analysis as a sparse recovery problem. Hence, even a sub-Nyquist sampling of the interferogram may provide enough information to successfully solve an under-determined system of linear equations to uncover the modal structure. Next, we introduce our approach in both the 1-D and 2-D cases.

A. One dimensional case

Consider an optical beam consisting of at most $s$ out of $N$ modes. According to (4), each interferogram sample, $P(\alpha_i), i = 1, ..., M$, can be written as an inner product

$$P(\alpha_i) - 1 = \langle \Phi_i, c \rangle; \ i = 1, ..., M,$$

(8)

where $c$ is an $N \times 1$ $s$-sparse vector containing the squared magnitudes of the modal coefficients and $\Phi_i$ is a row vector whose $n$-th element is $\cos(n\alpha_i), n = 1, ..., N$. In matrix form,

$$y = \Phi c,$$

(9)

where $y$ denotes an $M \times 1$ vector of $M$ interferogram samples, and $\Phi$ an $M \times N$ measurement matrix with rows $\{\Phi_i\}, i = 1, ..., M$. The numerical results in Section IV show
that if the phase shifts are chosen randomly, the measurement matrix $\Phi$ preserves most information required to recover the modal contents when $M << N$. Therefore, we adopt a sparse reconstruction algorithm, instead of FT, to recover the modal coefficients.

To account for the non-ideal nature of the interferometer setting, we rewrite the compressive interferometry measurements as

$$y = \Phi c + n,$$

(10)

where $n$ is an additive $M \times 1$ noise vector.

### B. Two dimensional case

In the case of two different DoFs, the problem of modal analysis can be posed as a sparse recovery problem with an under-determined system of equations. For the case where two GPOs are placed in two different arms, we rewrite the interferogram in (6) as,

$$P(\alpha_i, \beta_j) = 1 - \langle \Phi_{i,j}, c \rangle; \quad i = 1, \ldots, M_1; \quad j = 1, \ldots, M_2,$$

(11)

where $P(\alpha_i, \beta_j)$ is the interferogram sample corresponding to phase shifts $\alpha_i$ and $\beta_j$ in the two DoFs. Note that $\Phi_{i,j}$ is an $1 \times N (N = N_1 \times N_2)$ row vector with elements $\cos(n \alpha_i - m \beta_j), n = 1, \ldots, N_2; m = 1, \ldots, N_1$. Also, $c$ is an $N$-dimensional sparse vector containing the squared magnitude of the modal coefficients. In matrix form, the compressed interferometry model is given by,

$$y = \Phi c + n,$$

(12)

where $y$ is an $M \times 1$ vector with $M = M_1 \times M_2$. The matrix $\Phi$ is an $M \times N$ matrix with rows $\{\Phi_{i,j}\}$ and $n$ is an $M \times 1$ random vector which models the sampling tolerance. Similarly, the interferograms in (7) can be modeled as in (12) by replacing $\cos(n \alpha_i - m \beta_j)$ with $\cos(n \alpha_i + m \beta_j)$ in the construction of $\Phi_{i,j}$. Again using an appropriate reconstruction algorithm, the modal coefficients can be recovered using $M << N$ samples.

### IV. EXAMPLES AND NUMERICAL RESULTS

In this section, we illustrate the performance of the proposed compressive interferometry approach in 3 different examples. In all examples the sampling noise is modeled as an additive white Gaussian noise vector $N(0, \sigma_n^2 1)$, where

$$\sigma_n^2 = \left( \frac{\|y\|_2^2}{M} \right) / 10^{\text{SNR}}$$

where SNR denotes the signal to noise ratio in dB. The modal recovery error is defined as a normalized distance between the true vector $c$ and the recovered vector $\hat{c}$, i.e.,

$$\frac{|c - \hat{c}|_2^2}{|c|_2^2}.$$  

The uniform distribution $U[-\pi, \pi]$ is adopted to generate the phase shifts used to the interferometric samples. For reconstruction, we use the Smoothed $\ell_0$-norm (SL0) algorithm.

#### A. 1-D: Temporal Harmonics

We consider a basis consisting of $N = 100$ discrete temporal harmonics with main angular frequency $\omega_0 = 2\pi \times 3 \times 10^{11} \text{ rad/s}$ as $\{e^{-j m \omega_0 t}\}_m$. We construct an optical waveform as in (1) with $s = 4$ non-zero coefficients. Fig. 1 shows the recovery error of the proposed compressive approach for $\text{SNR} = 20 \text{ dB}$ and $\text{SNR} = 30 \text{ dB}$. As shown, a good performance is attained using around $M = 30$ samples. To compare the performance of the proposed approach to the one using FT at Nyquist rate, we set a very low main angular frequency $\omega_0 = 2\pi$. Fig. 2-(a) shows that at high SNR both approaches achieve a small error, however, in the case of FT the phase shifts should be necessarily chosen with a very small step size which is difficult to achieve in practice, specially at high frequencies. This comparison also could be seen in Fig. 2-(b).

#### B. 2-D: Temporal and Spatial Harmonics

In this section, we consider $N_1 = 10$ temporal harmonics with main angular frequency $\omega_0 = 2\pi \times 3 \times 10^{11} \text{ rad/s}$ and $N_2 = 10$ spatial harmonics with parameter $k = \omega_0/c$ (where $c$ is the speed of light in space) for two bases. The results in Fig. 3-(a) show that a successful recovery is achievable from about $M = 35$ samples. This Figure also illustrates that the recovery error of the non-zero elements is almost the same as that of the whole vector. This implies that the recovery error of the zero elements is low even with smaller $M$. Thus, if only support recovery is contemplated, the proposed algorithm could even lead to further complexity reduction. Fig. 3-(b) and Fig. 4 illustrate the performance in this case for several spatial examples. The intensity profiles shown in Fig. 3-(b) are calculated as $I = |E(x, y)|^2$.

#### C. 2-D: OAM and LG Modes

In this example, we consider a field containing OAM and LG modes to illustrate the flexibility of the proposed approach in arbitrary DoFs. Thus, the radial coordinate $\rho$ and the angular
coordinate $\theta$ are the DoFs, and $\ell$ and $p$ the corresponding modal indices. Two GPOs are used in the same arm of an interferometer; one GPO is a spatial rotator for the angular DoF, and the other is a fractional Hankel Transform (fHT) corresponding to the radial DoF. As demonstrated in Fig. 5-(a), a small reconstruction error is achievable using only $M = 35$ measurements. Three different cases with three different sets of non-zero coefficients are examined and the recovered coefficients are illustrated in Figures 5-(b). Successful modal analysis is clearly achieved with only $M = 35$ measurements. Since the measurement matrix is not precisely random, some variations in the recovery performance with different phase shifts are expected. However, as seen in Fig. 5-(b), these variations do not influence the consistency of the proposed algorithm.

V. CONCLUSION

In this paper, we proposed a compressive interferometry approach for optical modal analysis in arbitrary DoFs. First, the interferogram is modeled as a sparse linear system with a measurement matrix corresponding to delay settings of GPOs. Then, we used a sparse reconstruction algorithm to recover the modal coefficients. It was shown that efficient recovery of the modal content is achievable even from a very small number of measurements. Hence, the proposed approach can significantly reduce the computational complexity and mitigate the practical difficulties associated with interferometry-based modal analysis. Therefore, this approach holds potential to enhance real time processing in optical imaging and communications.

REFERENCES