Integer codes and lattice packings by cubes of sidelength \(k\)

Ulrich Tamm

**Abstract**— Integer codes correcting a single error in the maximum metric are considered. This corresponds to a packing of tori by cubes. For an asymmetric error of size one these cubes have side length 2 and the problem can be shown to be equivalent to finding zero-error codes for cycles in the sense of Shannon and Lovasz. For side length greater 3 the equivalence of single error correcting integer codes and zero-error codes does not hold any more.

**Index Terms**— integer codes, zero-error capacity, packing by cubes

I. INTRODUCTION

Integer codes correcting a single error of limited magnitude in one component as discussed in [12] have applications, for instance, in flash memories [15] or in peak-shift corrections for run-length-limited coding [9] or even in steganography [10]. A geometrical analysis is possible via lattice packings of the \(n\)-space by certain star bodies, which are denoted by Stein [18], [19] as cross and semicross. Perfect codes then correspond to tilings, for which a powerful algebraic tool, namely group factorizations, is a very helpful instrument, e.g., [14], [20], [21].

In the discussion of the results of [13] concerning the construction of single-error correcting codes on the \(a \times a\)-grid Morita asked what would happen if also the direct diagonal neighbours would be included into the error spheres. Although the resulting error-sphere – a cube or a box – seems to be much simpler then the cross or semicross an exact answer to Morita’s question would be very difficult to find when naturally the \(a \times a\) grid is extended to a torus.

In this case the \((1,2)\)-semicross would be extended to a \(2 \times 2\) cube and the \((1,2)\)-cross would be extended to a \(3 \times 3\) cube. A single-error correcting code then would correspond to a packing of the \(a \times a\) – torus by \(2 \times 2\) – cubes for asymmetric errors (the semicross) and by \(3 \times 3\) – cubes for symmetric errors (the cross). The generalization to higher dimensions is obvious.

Golomb [5] back in 1969 had already systematically analyzed the error spheres arising from several metrics in terms of unit cubes whose centers are points in the integer lattice. The cross and the semicross, which later turned out to be important for integer codes, he denoted as Stein sphere and Stein corner, respectively. Further, the Hamming sphere consists of a whole dimension, hence of infinitely many cubes and the Lee sphere has a very complicated structure. The tilings of \(R^n\) by the cross of armlength 1 led Golomb and Welch [6] to the complete characterization of perfect Lee codes of distance 1.

Golomb [5] points out that the Hamming sphere corresponds to the \(L_0\)-metric and the Lee sphere to the \(L_1\)-metric. The error spheres corresponding to the \(L_\infty\)-metric he denotes as the Shannon spheres, “since they generalize a coding problem considered by Shannon”. This is the famous zero-error capacity problem for odd cycles [16], which can equivalently be formulated as the problem of packing an \(a \times a \times \ldots \times a\)–torus by \(2 \times 2 \times \ldots \times 2\)–cubes in \(n\) dimensions. Of course, the problem is only interesting for odd \(a\), since for even \(a\), trivially, a tiling exists. In Section 2 we shall investigate a little further the equivalence between certain integer codes and the zero-error capacity.

II. INTEGER CODES AND ZERO-ERROR CAPACITY

As described in the introduction the connection of zero-error capacity problems to codes correcting errors in the Shannon sphere has been mentioned already by Golomb [5]. However, he did not provide a bijection to the zero-error codes in products of cycles. Actually, the example provided, namely the \(3 \times 3 \times 3\) cube is even a little misleading. So we shall discuss this bijection now.

In two dimensions, the corners of the unit cubes forming the \(2 \times 2\) cube are the vertices in the graph – here corresponding to the strong product of a path of length 2 – with the central node of the path representing a vertex on the odd cycle and the final nodes of the path representing the neighbours with which the vertex can be confused.

On the other hand, the inclusion of the diagonal neighbour in a \((1,2)\) - semicross (or Stein corner) would also yield a \(2 \times 2\) cube as an error sphere. Thus, a packing of an \(a \times a\) torus by \(2 \times 2\) cubes would correspond to a single–error correcting code in the Shannon sphere (or to an integer code correcting a single asymmetric error of amplitude 1 in one component) and also to a zero-error code for the strong product of cycles of length \(a\) such that these two problems are equivalent. The generalization to higher dimensions is straightforward.

This geometrical approach to the zero-error capacity of odd cycles was used in several papers ([17], [1], [17]) before Lovász [11] determined the zero-error capacity of the \(C_5\) via algebraic graph theory. Later Bohman [2] made use of such a construction from [1] in order to derive some bounds for cycles of large length.

III. PACKING BY BIGGER CUBES

Due to its importance for the zero–error capacity in literature mostly packings of tori by \(2 \times 2 \times \ldots \times 2\)–cubes have been considered. A \(3 \times 3\)–cube then would correspond to a strong product of a path of length 3 which consists of 4 vertices.
From such a path the confusability condition for the zero-error problem is not so easily constructed, simply, since there is no central node in the path. Thus, a generalization to a zero-error capacity is a little problematic: what will be the underlying graph and the confusability condition?

However, for integer codes, the Shannon sphere consisting of $3 \times 3$ cubes arises naturally, when the direct neighbours of the cross (or Stein sphere) are included in the error sphere. This corresponds to codes correcting a single symmetric error of amplitude 1 in one component. So there is a natural application also for the problem of packing a $a \times a \times \ldots \times a$ cubes by $3 \times 3 \times \ldots \times 3$ cubes.

Crosses and semicrosses with armlength $> 1$ would yield $k \times k \times \ldots \times k$ cubes as Shannon spheres (or master spheres) as in [5]. Hence, a natural extension of the packing problem of an $a \times a \times \ldots \times a$ torus by $k \times k \times \ldots \times k$ cubes in this direction is possible.

Probably because of the difficulties of an extension of the zero-error capacity of odd cycles in this packing problem generalizations to larger sidelength $k$ of the cubes have not been discussed very intensively in literature. In [4] coverings of tori by $k \times k$–cubes for $k > 2$ are constructed. It is pointed out that in Hales’ PhD thesis [7] also such packings are considered, however, in the resulting paper [8] they are seemingly not included.

For a divisible by $k$ trivially a tiling exists by placing the cubes adjacent to each of its neighbours. This construction is not necessarily optimal any more if $a$ is not divisible by $k$.

The resulting packing may be improved as, for instance, via the construction in [22] based on Shannon’s idea for $k = 2$ and $a = 5$:

For $a = k^2 + 1$ an $a \times a$ torus can be optimally packed by $k \times k$ cubes by placing the lower left corners in the positions $(i,j)$ with $i + kj = 0 \mod k^2 + 1$.

This yields a packing with $k^2 + 1$ cubes hereby improving on the trivial packing above which consists only of $k^2$ cubes. Further similar constructions in dimension 2 can be obtained.

Another problem is the derivation of upper bounds on the size of a packing. Because the equivalence to a graph theoretic problem breaks down for $k > 2$ the Lovasz’ bound relying heavily on the eigenvalues of the underlying graph adjacency matrix cannot be applied any more.

IV. Concluding remarks

Integer codes correcting a single error in the maximum metric are considered. The equivalent problem of packing a torus by $k \times k \times \ldots \times k$ cubes had already been addressed by Golomb who denoted the error spheres arising from theses cubes as Shannon sphere and master sphere.

The name Shannon sphere was already motivated by Shannon’s zero-error capacity problem for odd cycles. Indeed, for $k = 2$ the two problems – single error correcting codes in the maximum metric an zero-error capacity for cycles – are equivalent. Lower bounds can, of course be obtained constructively. Lovasz’ famous bound via algebraic graph theory for odd cycles, hence, also yields an upper bound on the size of integer codes correcting a single asymmetric error.

One might also ask for a single symmetric error which would correspond to a packing of a torus by $3 \times 3 \times \ldots \times 3$ cubes. Here the Lovasz bound does not seem to work any more because first an equivalent graph theoretic problem has to be found.

References


