Some theoretical limits on nuclear source localization and tracking

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Abstract—We consider the simple and general estimation problem of finding the location of a nuclear source from radiation measurements. Our objective is to study the effect of the inherent quantum randomness of radioactive emissions on the accuracy to which nuclear sources can be localized. To this end, we consider an ideal mobile detector making perfect, noiseless measurements and formulate a general problem of maximum likelihood estimation of source location using such measurements. For the case of a stationary source and a detector moving with uniform speed in a straight line, we derive solutions to the maximum likelihood location estimate as well as the corresponding Cramer-Rao lower bounds. We present a simple iterative procedure for calculating the ML estimate, and argue that in the asymptotic case of source strength becoming large, the procedure converges to the ML estimate with high probability and this estimate is unbiased. We also present simulations showing that the maximum likelihood estimates approach the Cramer-Rao bounds, and comment on the implications of these theoretical results with ideal detectors and perfect estimators to the problem of nuclear source localization.

Index Terms—Nuclear source localization, maximum likelihood estimation, inhomogenous Poisson process.

I. INTRODUCTION

Identification and tracking of nuclear materials is a problem of obvious importance, and has attracted a great deal of engineering effort [1], [2], [3], [4] in the recent past. The technical challenges of this problem are compounded by the fact that nuclear materials that are of interest vary widely in their concentrations and compositions [4], and are often concealed with shielding material, and buried in background radiation that is always present, but varies widely in strength and spectrum over time and space [5]. In addition, there are also many types of detectors [6] and sensors used in practical tracking systems, and these too vary widely in their quality and sensitivity.

However, on top of all these practical challenges, there is also a theoretical limitation arising from the fact that nuclear radiation is a fundamentally probabilistic physical process, and therefore there is an irreducible amount of randomness and uncertainty associated with radiation measurements. Specifically, radiation from a source consists of discrete emissions of particles, and likewise detection of radiation fundamentally consists of a sequence of events involving the absorption of discrete particles. These emission and detection events are modeled statistically as Poisson arrival processes [7], [8].

While the variability of radiation measurements because of this random Poisson statistics is well-known and routinely taken into account in the literature [1] on nuclear source detection and localization, the effect of this randomness has never been studied separately from other uncertainties like background radiation and modeling errors. While the latter effects are often larger in real-world nuclear detection applications, the variability of the Poisson process itself can have nonnegligible effects, especially with weak sources, inexpensive detectors and fast-moving sources and/or detectors [2] that may limit observations intervals to be short. Indeed, to our knowledge, the only previous work to study the effect of the randomness in the nuclear detection process separately is [9] which looks at the probability of detection error and false alarm resulting from the randomness of the emissions.

In this paper, we examine the theoretical limits on the achievable accuracy of localization of nuclear sources imposed by the inherent randomness of nuclear emission and detection process itself, even in the absence of any other external impairments in the detection process such as measurement errors, background noise, modeling uncertainties and so on. This study is intended as a step towards better understanding of the effect of this randomness. In addition, our theoretical limits with ideal sensors represent upper bounds on the performance achievable with any real-world detector, and provide important benchmarks for practical detection.

The rest of this paper is organized as follows. Section II presents our formulation of the general localization problem of a nuclear source with an ideal detector, as well as the simple special case of a detector moving in a straight line with uniform speed. The maximum likelihood solution to the localization problem and the corresponding Cramer-Rao lower bounds are derived in Section III. Section IV presents numerical simulation results and Section V concludes.

II. PROBLEM FORMULATION

We consider first the general problem of estimating the trajectory of a moving nuclear source using the measurements of a mobile, ideal detector. We then specialize to a stationary source and a detector moving in a straight line with uniform speed and a simple square-law signal strength model. While much of our analysis in this paper can be extended to the more general problem, we focus on these special cases as they
admit elegant closed form expressions that are revealing and insightful.

A. The ideal nuclear detector

First, we define the characteristics of our ideal detector. It is a device that records the precise time of arrival of every particle absorbed by its detection hardware. We ignore possible limits on the timing accuracy of such a detector due to the quantum energy-time uncertainty principle [10].

This ideal device is of course strictly superior to any practical nuclear detector in the sense that the observations of any practical device can be represented as a (possibly noisy) function of the measurements of the ideal sensor. For instance, a detector that uses average arrival rates for signal strength measurements can equivalently be represented as averaging interarrival times from an ideal sensor. Similarly, a practical sensor that detects particle counts over a set of finite observation intervals can be represented as a device that quantizes an ideal sensor’s time-stamp measurements. Designs for practical sensors with detection time bins as short as 150 ms have been reported [11]. As detection intervals become shorter, such practical detectors can approach the performance of an ideal sensor.

B. General Problem statement

Consider a nuclear source moving along a trajectory \( z_0(t) = [x_0(t), y_0(t)] \) parameterized by an arbitrary set of unknown variables \( \theta \), and a detector moving along the known trajectory \( z(t) \). The distance of the detector from the source at some time \( t \) is \( d(t) = ||z_0(t) - z(t)|| \) where \( || \cdot || \) denotes the 2-norm representing the usual Euclidian distance.

In the sequel, we assume that the mean arrival rate of particles from the source depends only on the distance from the source. In other words, we do not consider directional detectors such as CZT Compton scattering devices [12] and limit ourselves to scintillation-type devices [13] and isotropic media. However, our analysis is not limited to this case, and can be easily extended. Accordingly, in our formulation, particles from the source are absorbed by the detector at discrete times representing an instance of an inhomogenous Poisson process [14] with the mean arrival rate \( \lambda(t) = f(d(t)) \), where \( f(d) \) is a decreasing function of distance representing the attenuation of the source strength over space.

The general problem of nuclear source localization can be stated as the problem of finding the source strength and trajectory by estimating the set of unknown parameters \( \theta \), given a (possibly countably infinite) set of observed arrival times \( \tau_m, m \in \{1, \ldots, n\} \) at the ideal detector over some (possibly infinite) observation interval \( (T_1, T_2) \). This problem is readily generalized to a network of many sensors.

C. Special case of a stationary source, and detector moving with uniform speed in a straight line

As stated earlier, in this paper, we mostly focus on a simple, special case of the problem in II-B. In this special case, the source is stationary at an unknown location \( z_0(t) = [x_0, y_0] \), and the detector is moving with a constant speed \( s \) in a straight line that without loss of generality can be taken to be the X-axis. Furthermore, we choose our coordinates so that the detector’s position at time \( t = 0 \) is chosen as the origin. Then the detector’s trajectory is \( \mathcal{L} : z(t) = [st, 0] \) and the corresponding distance from the source is \( d(t) = \sqrt{y_0^2 + (x_0 - st)^2} \). As observations are from a detector confined to a line, to remove the flip ambiguity in localization this engenders, we assume that \( y_0 > 0 \).

In this special case, we also assume that the signal attenuation follows a simple inverse square law \( f(d) = \frac{A_0}{d^2} \) which gives for the mean arrival rate at the detector

\[
\lambda(t) = \frac{A_0}{y_0^2 + (x_0 - st)^2}, \tag{1}
\]

where \( A_0 \) is a source strength parameter that depends on the type, shape and total volume of the nuclear source as well as the detector characteristics. This setup is depicted in Fig. 1.

The unknown parameters for this special case consists of the triplet \( \theta = [A_0, x_0, y_0] \). The total number of arrival times recorded by an ideal detector is a Poisson random variable \( N \) with mean equal to the area under the curve generated by \( \lambda(t) = f(d(t)) \), which turns out to be finite even for an infinite time horizon for the inverse-square law attenuation model in (1):

\[
E[N] = \int_{-\infty}^{\infty} \frac{A_0}{y_0^2 + (x_0 - st)^2} \, dt = \frac{\pi A_0}{s y_0} \tag{2}
\]

The nuclear source localization problem in this special case, is then the problem of estimating the triplet \( \theta = [A_0, x_0, y_0] \) given a set of measured time-stamps representing the arrivals of radiation particles \( \tau_m, m \in \{1, \ldots, n\} \) over the infinite time horizon \( t \in (-\infty, \infty) \). One realization of the random arrival process is shown in Fig. 2.

III. Maximum Likelihood Estimation of Nuclear Source Location and Strength

We now derive the likelihood function for the general estimation problem presented in Section II-B. We will then specialize this likelihood function for the special case in Section II-C and derive the maximum likelihood solution. We state key results without proof and omit detailed derivations for space constraints.
We then have for the likelihood function
\[ l(\theta) = \frac{1}{\Delta t} \int_{t_k}^{t_{k+1}} \lambda(t) dt \]

Then, the expected number of arrivals in \( I_k \) is \( n_k = \lambda_k \Delta t \). Note that (3) implies \( \lim_{\Delta t \to 0} \lambda_k = \lambda(t_k) \). The probability of receiving \( m \) arrivals in the interval \( I_k \) follows the Poisson distribution: \( p(m) = \frac{n_e^m e^{-n_e}}{m!} \). Assuming that \( \Delta t \) is chosen small enough, each of the intervals \( I_k \) contain either \( m = 0 \) or \( m = 1 \) arrivals with high probability. Furthermore, the number of arrivals in each interval are all independent random variables. Let \( n_i \) denote the interval which contains the \( i \)th arrival \( t_i \). Let \( N \) denote the set of intervals in which \( m = 1 \) arrivals occurred i.e. \( N = \{ k : \exists i \text{ such that } n_i = k \} \). Let \( N^c \) be the complementary set of intervals in which \( m = 0 \) arrivals occurred. Finally, note that (3) implies \( \lim_{\Delta t \to 0} \lambda_n = \lambda(t_i) \).

We then have for the likelihood function
\[ l(\theta) = f_{\tau|\theta}(\tau|\theta) = \prod_{k \in N} \lambda_k \Delta t e^{-\lambda_k \Delta t} \prod_{k \in N^c} e^{-\lambda_k \Delta t} \]
\[ = (\Delta t)^N \left( \prod_{k \in N} \lambda_k \right) \left( \prod_{k \in N^c} e^{-\lambda_k \Delta t} \right) \]

Noting that \( (\Delta t)^N \) does not depend on the unknown parameters \( \theta \), and taking limits as \( \Delta t \to 0 \) on the remaining terms, we can now write down a modified log-likelihood function
\[ L(\theta) = -\int_{-\infty}^{\infty} \lambda(t) dt + \sum_{i=1}^{n} \log (\lambda(t_i)) \]

Specializing (6) to the case of Section II-C gives:
\[ L(A, x, y) = -\frac{\pi A}{sy} + n \log A - \sum_{i=1}^{n} \log (y^2 + (x - s \tau_i)^2) \]

B. The Maximum Likelihood estimate

For a given set of observations by the detector \( \tau = [\tau_1, \tau_2 \ldots \tau_n] \), the maximum likelihood estimate is defined as:
\[ \hat{A}, \hat{x}, \hat{y} = \arg \max_{A>0, x,y>0} L(A, x, y). \]

The following function will appear repeatedly in the sequel:
\[ \alpha(x, y, t) = \frac{1}{y^2 + (x - st)^2} \]

Comparing with (1), we note \( \alpha(x, y, t) \) is simply the arrival rate of particles due to a fictional source with unit strength and location \( (x, y) \) at our ideal moving detector at time \( t \).

We now state several properties of the ML estimate.

**Proposition 1.** The log-likelihood function in (7) has a unique critical point and this point is the global maximum for \( L(A_0, x_0, y_0) \).

Differentiating \( L(A_0, x_0, y_0) \) with respect to \( A_0, x_0, y_0 \) and equating to zero immediately gives for the ML estimates:
\[ \hat{A} = \frac{nsy}{\pi}, \]
\[ \sum_{i=1}^{n} \alpha(\hat{x}, \hat{y}, \tau_i) (\hat{x} - s \tau_i) = 0, \]
\[ \sum_{i=1}^{n} \alpha(\hat{x}, \hat{y}, \tau_i) = \frac{n}{2\hat{y}^2} \]

Equation (10) provides an explicit expression for \( \hat{A} \), while (11, 12) provide a pair of equations that implicitly determine \( \hat{x}, \hat{y} \).

Now given constant \( a \), define
\[ Q(y) = y^2 \sum_{i=1}^{n} \alpha_i(a, y) - \frac{n}{2} \]
\[ = \sum_{i=1}^{n} \frac{y^2}{y^2 + (a - s \tau_i)^2} - \frac{n}{2} \]

Clearly, \( Q(0) = -\frac{n}{2} \), and \( \lim_{y \to \infty} Q(y) = \frac{n}{2} \). Since \( Q(y) \) is a strictly monotonic function of \( y \) for \( y > 0 \), it follows then that there is a unique positive solution to \( Q(y) = 0 \). Let us denote this solution \( y = f_1(a) \). We then have the following result:

**Proposition 2.** The sequence \( y[m] \) defined by the recursion \( y[m+1] = \sqrt{2 \sum_{i=1}^{m} \alpha(a, y[m], \tau_i)} \) converges to \( f_1(a), \forall a \in \mathbb{R} \).

Comparing (14) with (12), we see that the ML solution satisfies \( \hat{y} = f_1(\hat{x}) \). Thus Proposition 2 provides a simple...
recursive procedure to calculate $\hat{y}$ given an estimate of $\hat{x}$. Now define

$$f_2(x) = \sum_{i=1}^{n} s_i \alpha(x, f_1(x, \tau_i))$$

(15)

Comparing (15) with (11), we see that $\hat{x} - f_2(\hat{x}) = 0$. We finally have the following proposition.

**Proposition 3.** The sequence $x[m]$ defined by the recursion $x[m+1] = f_2(x[m])$ converges to $\hat{x}$.

Propositions 3 and 2 along with the expression for $\hat{A}$ in (10) provide a complete numerical procedure to calculate the maximum likelihood solution to our localization problem.

Finally, we can show that the inverse of the Fisher Information Matrix for the likelihood function is

$$FIM^{-1} = \begin{bmatrix}
\frac{3\pi y_0 A_0}{\pi} & 0 & \frac{2\pi y_0^2}{\pi} \\
0 & \frac{2\pi y_0^2}{\pi A_0} & 0 \\
\frac{2\pi y_0^2}{\pi} & 0 & \frac{2\pi y_0^2}{\pi A_0}
\end{bmatrix}$$

(16)

Let $N_0 = E[N]$ denote the expected number of total particle arrivals at the ideal detector. We know from (2) that $N_0 = \frac{\pi A_0}{\pi y_0}$. The expression for the inverse of the Fisher Information Matrix in (16) can be rewritten as:

$$FIM^{-1} = \frac{1}{N_0} \begin{bmatrix}
3A_0^2 & 0 & 2A_0y_0 \\
0 & 2y_0^2 & 0 \\
2A_0 y_0 & 0 & 2y_0^2
\end{bmatrix}$$

(17)

We can now infer several properties of this estimation problem, some of which are intuitively obvious, while others are somewhat unexpected.

1) The Cramer-Rao lower bound for the variances of $[A_0, x_0, y_0]$ can now be written as:

$$\sigma_{A,\min}^2 = \frac{3A_0^2}{N_0}, \sigma_{x,\min}^2 = \frac{2y_0^2}{N_0}$$

(18)

2) Equation (18) shows that the bounds are completely independent of $x_0$. This makes sense from the geometry of the problem: the error bounds only depend on the distance of closest approach of the detector to the source which is simply $y_0$ and does not depend on $x_0$.

3) Note also that the normalized bounds $\frac{\sigma_{A,\min}^2}{A_0^2}$, $\frac{\sigma_{x,\min}^2}{y_0^2}$ and $\frac{\sigma_{y,\min}^2}{y_0^2}$ are all inversely proportional to $N_0$. Thus we get better source location estimates with larger $N_0$. This can result of larger $A_0$ i.e. stronger sources can be better localized. The same result can also be achieved using a detector with greater sensitivity or larger aperture or more interestingly by decreasing the speed $s$ of the moving detector so it collects more samples on average.

4) Finally, the fact that the bounds on the variance of $x_0$ and $y_0$ are identical is a mild surprise, given that there is no obvious symmetry between the $X$- and $Y$-coordinates. We are not aware of any deeper significance to this coincidence, but as we show below, it is indeed confirmed by numerical simulation results.

**Remark on proofs.** While our numerical studies indicate that Propositions 1, 2 and 3 are generally valid for arbitrary set of observations $\tau$, a general proof seems difficult. However, in the asymptotic case where the source strength $A_0$ is large enough, or the detector speed $s$ is small enough that the number of particles detected $n$ is large, we can use law of large number arguments to construct simple proofs. Thus, for instance, we can show that there are unique solutions to (10), (11) and (12) under the assumption:

$$\frac{1}{n} \sum_{i=1}^{n} \alpha(x, y, \tau_i) \approx \frac{1}{N_0} E \left( \sum_{i=1}^{n} \alpha(x, y, \tau_i) \right)$$

(19)

$$\equiv \frac{1}{N_0} \int_{-\infty}^{\infty} \alpha(x, y, t) \lambda(t) dt, \forall x, y$$

(20)

Similar considerations can be used to show the fixed point results in Propositions 2 and 3. We defer details to a future publication.

**IV. SIMULATION RESULTS**

We now present some numerical results where we compare the ML estimate with the Cramer-Rao bounds. The Poisson process in (1) is simulated using standard methods [15],[14] with the source location $z_0 = [0, 500]^T$ m and the detector speed $s = 20 m s^{-1}$. The mean squared error estimation error of the ML estimate is calculated for different values of $A_0 \pi/(sy_0)$; $s$ is kept constant while varying $A_0$. The results averaged over 10000 independent realizations of the Poisson arrival process is shown in Fig. 4.

Similar results are obtained if we vary the other parameters $s$ and $x_0$, $y_0$ as predicted by (16). The main takeaway from these simulations is that the variance of the ML estimate quickly converges to the CRLB when the expected number of arrivals exceeds 50 or so.

**V. CONCLUSION**

In this paper, we made a start towards exploring the fundamental limits on the achievable accuracy of localizing nuclear sources because of the inherent randomness of nuclear
emission and detection processes. While we considered the special case of a stationary source and an ideal detector moving in a straight line with uniform speed under an inverse square law attenuation model, the analysis can be extended and generalized to tracking moving sources using a network of detectors and taking into account shielding materials as well as practical detector characteristics.

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REFERENCES