Globally Coupled LDPC Codes

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Abstract—This paper presents a special type of LDPC codes with a structure related to but different from that of the spatially coupled LDPC codes. For an LDPC code of this type, its Tanner graph is composed of a set of small disjoint Tanner graphs which are connected together by a group of overall check-nodes, called global check-nodes. Codes of this type are called globally coupled LDPC codes and they perform well over both the additive Gaussian white noise and the binary-erasure channels. Furthermore, they are very effective at correcting erasures clustered in bursts. A two-phase local/global iterative scheme allows correction of local random and global errors and/or erasures in two phases.

I. INTRODUCTION

In this paper, we present a special type of LDPC codes with a structure related to but different from that of the spatially coupled (SC) LDPC codes [1], [2], [3]. For an LDPC code of this type, its Tanner graph is composed of a set of small disjoint Tanner graphs which are connected together by a group of overall check-nodes (CNs), called global CNs. A code of this type is called a CN-based globally coupled (GC) LDPC code.

Two algebraic methods are presented for constructing GC-based quasi-cyclic (QC) LDPC codes. Both code construction methods are based on dispersing (or expanding) the entries of a base matrix over a nonbinary (NB) finite field into circulant permutation matrices (CPMs) and/or zero matrices (ZMs) of the same size. Examples show that the codes constructed perform well on both the additive white Gaussian noise channel (AWGNC) and the binary erasure channel (BEC). Codes constructed by one method are effective for correcting erasures clustered in bursts. Also presented in this paper is a two-phase local/global iterative scheme for decoding CN-based GC-LDPC codes. The decoding scheme allows correction of local random and global errors and/or erasures.

The rest of the paper is organized as follows. Section II presents two classes of base matrices over NB finite fields which satisfy a certain constraint. Based on these two classes of base matrices, we can construct QC-GC-LDPC codes whose Tanner graphs have girth at least 6. Section III presents the first method for constructing CN-based QC-GC-LDPC codes. Section IV presents an iterative scheme for decoding CN-based GC-LDPC codes in two phases, local then global. Section V presents the second method which is devised specifically for constructing CN-based QC-GC-LDPC codes which are capable of correcting erasures clustered in bursts. Section VI concludes the paper with some remarks.

II. BASE MATRICES, MATRIX-DISPERSION AND QC-LDPC CODES

Most of the algebraic constructions have several important ingredients, including base matrices, matrix-dispersion (or matrix-expansion), and masking [4]. By a proper choice and combination of these ingredients, algebraic LDPC codes with excellent overall error performance can be constructed. Algebraic LDPC codes, in general, have much lower error-floors than randomly constructed LDPC codes. In this section, we present two structured base matrices over NB finite fields for constructing CN-based QC-GC-LDPC using the matrix dispersion construction [4].

Let GF(q) be an NB field with q elements, where q is a power of a prime. Let α be a primitive element of GF(q). Then, the powers of α, α0 = 1, α, α2, . . . , αq−2, give all the nonzero elements of GF(q) and αq−1 = α0 = 1. Form the following (q − 1) × (q − 1) matrix over GF(q):

\[ B_1 = \begin{bmatrix}
α^0 & 1 & α & α^2 & \cdots & α^{q-3} & α^{q-2} & 1 \\
α^{q-2} & 1 & α & α^2 & \cdots & α^{q-4} & α^{q-3} & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
α & α^2 & α^3 & \cdots & \cdots & α^{q-2} & α^{q-1} & 1
\end{bmatrix}. \]

(1)

From (1), we see that the matrix \( B_1 \) is a circulant matrix. This circulant matrix has the following structural properties: 1) All the entries in a row (or a column) are distinct elements in GF(q); 2) Any two rows (or two columns) differ in every location; and 3) There are exactly \( q-1 \) zero entries and they lie on the main diagonal of the matrix.

A very important structural property of \( B_1 \) is that any \( 2 \times 2 \) submatrix is nonsingular. To prove this structural property, we consider a \( 2 \times 2 \) submatrix \( M \) of \( B_1 \). It follows from the cyclic structure of \( B_1 \) that \( M \) must be of the following form:

\[ M = \begin{bmatrix}
α^i & 1 \\
α^{-j} & α^{-j-k}
\end{bmatrix}, \]

where \( 0 \leq i, j, k < q-1 \) and \( i < j \). The determinant of this \( 2 \times 2 \) submatrix \( M \) of \( B_1 \), after simplification, is equal to \( (α^j - α^i)(1 - α^{q-1-k}) \). Since \( α^j \neq α^i \) and \( α^{q-1-k} \neq 1 \), the term \( (α^j - α^i)(1 - α^{q-1-k}) \) cannot be zero. Hence, the
2 × 2 submatrix $M$ is nonsingular. The property that any 2 × 2 submatrix of $B_1$ is nonsingular is called the 2 × 2 submatrix (SM) constraint and the matrix $B_1$ is called a 2 × 2 SM-constrained matrix.

The 2 × 2 SM-constraint ensures that the Tanner graph of a QC-LDPC code constructed based on any submatrix of $B_1$ using CPM-dispersion has a girth of at least 6 [5]. Using $B_1$ as the base matrix to construct QC-LDPC codes was first proposed in [6].

Now, we present another 2 × 2 SM-constrained matrix constructed based on the finite field $GF(q)$. Let $p$ be the largest (or any) prime factor of $q - 1$ and let $q - 1 = pe$. If $q - 1$ is prime, then $p = q - 1$. Let $\beta = \alpha^e$. Then, the order of $\beta$ is $p$, i.e., $\beta^p = 1$ and the elements, $1, \beta, \beta^2, \ldots, \beta^{p-1}$, form a cyclic subgroup $S$ of the cyclic group of $GF(q)$. We form the following $p \times p$ matrix:

$$B_2 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \beta & \beta^2 & \cdots & \beta^{p-1} \\ 1 & \beta^2 & (\beta^2)^2 & \cdots & (\beta^2)^{p-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \beta^{p-1} & (\beta^{p-1})^2 & \cdots & (\beta^{p-1})^{p-1} \end{bmatrix}.$$  \(2\)

All the entries of $B_2$ are elements of the cyclic subgroup $S$, which are all nonzero. Next, we prove that $B_2$ satisfies the 2 × 2 SM-constraint. To prove this, we label the rows and columns of $B_2$ from 0 to $p - 1$. For $0 \leq i, j, s, t < p$ with $i < j$ and $s < t$, we consider the following 2 × 2 matrix $M$ of $B_2$:

$$M = \begin{bmatrix} (\beta^i)^s & (\beta^i)^t \\ (\beta^j)^s & (\beta^j)^t \end{bmatrix}.$$  \(3\)

The determinant of $M$ is $\beta^{is+jt} - \beta^{it+js}$. If $\beta^{is+jt} = \beta^{it+js}$, the order of $\beta$ is $\beta^{(j-i)(t-s)}$ must be equal to 1. Since both $j - i$ and $s - t$ are positive integers less than $p$ and $p$ is a prime (the order of $\beta$), $\beta^{(j-i)(t-s)}$ cannot be equal to 1. Hence, $\beta^{is+jt} - \beta^{it+js} \neq 0$ and $M$ is nonsingular. Using $B_2$ as the base matrix to construct QC-LDPC codes was also first proposed in [6].

In many construction methods of binary LDPC codes, a nonzero element in $GF(q)$ is represented by a binary CPM. For $0 \leq j < q - 1$, we represent the element $\alpha^j$ by a binary CPM of size $(q - 1) \times (q - 1)$, denoted by $A(\alpha^j)$, whose generator has the unit-element “1” of $GF(q)$ as its single nonzero component at position $j$, where the columns and rows are labeled from 0 to $q - 2$. It is clear that the representation of the element $\alpha^j$ by the binary CPM $A(\alpha^j)$ of size $(q - 1) \times (q - 1)$ is unique and the mapping between $\alpha^j$ and $A(\alpha^j)$ is one-to-one. This matrix representation of $\alpha^j$ is referred to as the binary CPM-dispersion of $\alpha^j$. Note that the size of the binary CPM $A(\alpha^j)$ is the order of $\alpha$. The zero-element of $GF(q)$ is represented by a $(q - 1) \times (q - 1)$ zero matrix (ZM).

Let $\eta$ be a nonzero element in $GF(q)$ of order $p$, where $p$ is a proper factor of $q - 1$. The $p$ elements, $\eta^0 = 1, \eta, \eta^2, \ldots, \eta^{p-1}$, form a cyclic subgroup of order $p$ of $GF(q)$. For $0 \leq j < p$, we can represent the element $\eta^j$ by a binary CPM $A(\eta^j)$ of size $p \times p$ whose generator has its single nonzero component at position $j$. Clearly, the mapping between $\eta^j$ and $A(\eta^j)$ for $0 \leq j < p$ is one-to-one. In this case, we disperse an element of a cyclic subgroup of $GF(q)$ into a binary CPM of smaller size.

Suppose $GF(q) \setminus \{0\}$ is a cyclic group of a larger field $GF(q')$, i.e., $GF(q)$ is a subfield of $GF(q')$. Let $\zeta$ be a primitive element in $GF(q')$. Then, an element $\omega$ in $GF(q) \setminus \{0\}$ can be expressed as a power of $\zeta$, say $\omega = \zeta^\ell$, with $0 \leq \ell < q' - 1$. With respect to the larger field $GF(q')$, we can disperse $\omega$ into a binary CPM $A(\omega)$ of size $(q' - 1) \times (q' - 1)$ whose generator has its single nonzero component at location $\ell$. Again, the mapping between $\omega$ and $A(\omega)$ is one-to-one. In this case, an element of $GF(q)$ is dispersed into a binary CPM of larger size.

The above simply shows that an element of a field $GF(q)$ can be one-to-one dispersed into a CPM of size equal to, smaller than or larger than $q - 1$. This one-to-one CPM-dispersion of a field element will be used to construct QC-LDPC codes.

For $1 \leq m, n < q$, let $B_2(m, n)$ be an $m \times n$ submatrix of $B_1$. Since $B_1$ satisfies the 2 × 2 SM-constraint, $B_1(m, n)$, as a submatrix of $B_1$, must also satisfy the 2 × 2 SM-constraint. Disperse each nonzero entry of $B_2(m, n)$ into a binary CPM of size $(q - 1) \times (q - 1)$ as described above and each zero entry (if exists) into a ZM of size $(q - 1) \times (q - 1)$. This dispersion operation results in an $m \times n$ array $H_1(q - 1, q - 1)$ of CPMs and/or ZMs of size $(q - 1) \times (q - 1)$. The array $H_1(q - 1, q - 1)$ is called the binary CPM-dispersion of the matrix $B_2(m, n)$. $H_1(q - 1, q - 1)$ is an $m(n - 1) \times n(q - 1)$ matrix over $GF(2)$. Note that the array is a binary expansion of the NB matrix $B_2(m, n)$. The 2 × 2 SM-constraint property of $B_2(m, n)$ ensures that $H_1(q - 1, q - 1)$ satisfies the RC-constraint, i.e., no two rows (two columns) of $H_1(q - 1, q - 1)$ have more than one location where they both have nonzero entries [5, Corollary 1], [6]. The RC-constraint ensures that the Tanner graph associated with $H_1(q - 1, q - 1)$ has girth at least 6 [6], [7], [8], [9].

The null space of $H_1(q - 1, q - 1)$ gives a binary QC-LDPC codes of length $n(q - 1)$ whose Tanner graph has girth at least 6. The above construction of a QC-LDPC code is called CPM-dispersion construction. The matrix $B_1(m, n)$ is called the base matrix and the matrix $B_1$ is called the mother matrix.

Similarly, the 2 × 2 SM-constrained matrix $B_2$ given by (2) can also be used as a mother matrix. Let $B_2(m, n)$ be an $m \times n$ submatrix of $B_2$. Since the entries of $B_2$ are elements of a cyclic subgroup of order $p$ of the cyclic group $GF(q)$, we can either disperse $B_2(m, n)$ into an RC-constrained $m \times n$ array $H_2(q - 1, q - 1)$ of CPMs of size $(q - 1) \times (q - 1)$ or an RC-constrained $m \times n$ array $H_2(p, p)$ of CPMs of size $p \times p$. Since all the entries in $B_2$ are nonzero elements of $GF(q)$, $H_2(q - 1, q - 1)$ (or $H_2(p, p)$) consists of only CPMs. The null space of $H_2(q - 1, q - 1)$ (or $H_2(p, p)$) gives a QC-LDPC code of length $n(q - 1)$ (or $np$).
III. Construction of CN-Based QC-GC-LDPC Codes

In this section, we use \( B_1 \) to construct CN-based QC-GC-LDPC codes. Label the rows and columns of \( B_1 \) from 0 to \( q-2 \). Suppose \( q-1 \) can be factored as the product of two positive integers \( l \) and \( r \), i.e., \( br = q-1 \). Partition \( B_1 \) into \( r \) submatrices of size \( l \times (q-1) \), denoted by \( W_0, W_1, \ldots, W_{r-1} \), where for \( 0 \leq i < r \), the submatrix \( W_i \) consists of the \( il \)-th to the \( (i+1)l \)-th rows of \( B_1 \) (\( l \) consecutive rows of \( B_1 \)). Next, we partition the submatrix \( W_0 \) into \( r \) submatrices of size \( l \times l \), denoted by \( W_{0,0}, W_{0,1}, \ldots, W_{0,r-1} \), each consisting of \( l \) consecutive columns of \( W_0 \). Then, \( W_0 = \begin{bmatrix} W_{0,0} & W_{0,1} & \cdots & W_{0,r-1} \\ W_{0,r-1} & W_{0,0} & \cdots & W_{0,r-2} \\ \vdots & \vdots & \ddots & \vdots \\ W_{0,1} & W_{0,2} & \cdots & W_{0,0} \end{bmatrix} \) (3).

Each row-block (or each column-block) of \( B_1 \) is composed of the same set of constituent matrices \( W_{0,0}, W_{0,1}, \ldots, W_{0,r-1} \) in cyclic order. Since \( B_1 \) satisfies the 2 \times 2 SM-constraint, each of its constituent matrices \( W_{0,0}, W_{0,1}, \ldots, W_{0,r-1} \) also satisfies the 2 \times 2 SM-constraint. Furthermore, all these \( r \) constituent submatrices of \( B_1 \) satisfy the pair-wise (PW) 2 \times 2 SM-constraint. This simply follows from the block-cyclic structure of \( B_1 \) shown by (3).

For \( 0 \leq j < r \) and \( 1 \leq m, n < l \), we now take an \( m \times n \) submatrix \( R_{0,j} \) from \( W_{0,j} \). The submatrices \( R_{0,0}, R_{0,1}, \ldots, R_{0,r-1} \) are taken from \( W_{0,0}, W_{0,1}, \ldots, W_{0,r-1} \) under the following restriction: for \( j \neq j' \), the locations of the entries of \( R_{0,j} \) in \( W_{0,j} \) are identical to the locations of the entries of \( R_{0,j'} \) in \( W_{0,j'} \). Next, we form the following \( r \times r \) array \( R_1(m, n) \) of \( m \times n \) submatrices over GF(q) with a block-cyclic structure:

\[
R_1(m, n) = \begin{bmatrix}
R_{0,0} & R_{0,1} & \cdots & R_{0,r-1} \\
R_{r-1,0} & R_{0,0} & \cdots & R_{r-2,0} \\
\vdots & \vdots & \ddots & \vdots \\
R_{0,1} & R_{0,2} & \cdots & R_{0,0}
\end{bmatrix}
\] (4).

\( R_1(m, n) \) is a submatrix of \( B_1 \) and satisfies the 2 \times 2 SM-constraint. Since \( W_{0,0}, W_{0,1}, \ldots, W_{0,r-1} \) satisfy the PW-2 \times 2 SM-constraint, \( R_{0,0}, R_{0,1}, \ldots, R_{0,r-1} \) also satisfy the PW-2 \times 2 SM-constraint.

The CPM-dispersion of \( R_1(m, n) \) gives an \( rm \times rn \) array \( H_{1,q}(q-1, q-1) \) of CPMs and/or ZMs of size \((q-1) \times (q-1))\). The null space of \( H_{1,q}(q-1, q-1) \) gives a QC-LDPC code \( C_{1,q} \) of length \( rm(q-1) \) with rate at least \((n-m)/n \) whose Tanner graph has girth at least 6. Let \( v \) be a codeword in \( C_{1,q} \). This codeword can be divided into \( r \) sections in the form of \( v = (v_0, v_1, \ldots, v_{r-1}) \), each section consisting of \( n(q-1) \) components. If we cyclically shift \( v \) one section to the right, we obtain another codeword in \( C_{1,q} \). This simply follows from the block-cyclic structure of the base matrix \( R_1(m, n) \). We say that \( C_{1,q} \) has block-cyclic structure. We can also divide the codeword \( v \) into \( rn \) sections in the form \( v = (v_0, v_1, \ldots, v_{rn-1}) \), each section consisting of \( q-1 \) components. If we cyclically shift all the \( rn \) sections simultaneously one position to the right within the sections, we obtain another codeword in \( C_{1,q} \). In this case, we say that \( C_{1,q} \) has section-wise cyclic structure. This simply follows from the CPM-structure of the parity-check matrix \( H_{1,q}(q-1, q-1) \).

Therefore, the code \( C_{1,q} \) has a doubly quasi-cyclic structure.

In forming the \( r \times r \) cyclic array \( R_1(m, n) \) from the \( r \times r \) array \( B_1 \) of \( l \times l \) submatrices over GF(q) given by (3), there are \( l \times m \) rows in each row-block and \( l \times n \) columns in each column-block of \( B_1 \) which are unused. So, a total of \( r(l-1) \) rows of \( B_1 \) are not used in forming the array \( R_1(m, n) \). We denote the set of \( r(l-1) \) unused rows of \( B_1 \) by \( I \). For each row \( w \) in \( I \), we remove the components at the locations that correspond to the rows that are not used in forming the array \( R_1(m, n) \) from \( B_1 \). This results in a shortened row \( w* = (w_{0,0}, w_{0,1}, \ldots, w_{0,r-1}) \) which consists of \( r \) sections, each consisting of \( n \) components. The locations of the \( n \) components of the \( i \)-th section \( w_{0,i} \) correspond to the locations of the \( n \) columns of the submatrix \( R_{0,i} \) of the submatrix \( W_{0,i} \) in the array \( B_1 \). Let \( \Pi^* \) denote the set of \( r(l-1) \) shortened versions of the rows in \( \Pi \). The set \( \Pi^* \) consists of the locations of the rows in the first block-columns \( [R_{0,0}, R_{0,1}, \ldots, R_{0,r-1}] \) of \( R_1(m, n) \) that are disjoint.

Let \( s \) and \( t \) be two positive integers with \( 1 \leq s \leq r(l-1) \) and \( 1 \leq t \leq r \). Take \( s \) rows from \( \Pi^* \) and remove the last \( r-t \) sections of each row from the \( s \) chosen rows. With these \( s \) shortened rows, we form an \( s \times nt \) matrix \( X_{gc,cn}(s, t) \) over GF(q). Next, we form the following array over GF(q):

\[
R_{1,gc,cn}(m, n, s, t) = \begin{bmatrix}
R_{0,0} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
X_{gc,cn}(s, t) & \cdots & \cdots & R_{0,0}
\end{bmatrix}
\] (5).

The upper subarray of \( R_{1,gc,cn}(m, n, s, t) \) is a \( s \times t \) diagonal array with \( t \) copies of \( R_{0,0} \) on its main diagonal and the lower part is the \( s \times nt \) matrix \( X_{gc,cn}(s, t) \). The matrix \( R_{1,gc,cn}(m, n, s, t) \) is an \( (mt+s) \times nt \) matrix over GF(q) which is a submatrix of the matrix \( B_1 \). Since \( B_1 \) satisfies the 2 \times 2 SM-constraint, \( R_{1,gc,cn}(m, n, s, t) \) must also satisfy the 2 \times 2 SM-constraint. The subscripts “gc” and “cn” in \( X_{gc,cn}(s, t) \) and \( R_{1,gc,cn}(m, n, s, t) \) stand for “global coupling” and “check-node”, respectively.

Let \( G_{0,0} \) and \( G_{1,gc,cn} \) be the Tanner graphs of \( R_{0,0} \) and \( R_{1,gc,cn}(m, n, s, t) \), respectively. It follows from the structure of the matrix \( R_{1,gc,cn}(m, n, s, t) \) displayed by (5) that the Tanner graph \( G_{1,gc,cn} \) of \( R_{1,gc,cn}(m, n, s, t) \) is composed of \( t \) disjoint copies of the Tanner graph \( G_{0,0} \) of the matrix \( R_{0,0} \).
connected by $s$ global CNs that correspond to the $s$ rows of the matrix $X_{gc,cn}(s,t)$. The $s$ global CNs provide the only connections between any two disjoint copies of $G_{0,0}$, i.e., two disjoint copies of $G_{0,0}$ are only connected to each other through a set of global CNs.

Let $CPM(R_{0,0})$ and $CPM(X_{gc,cn}(s,t))$ denote the binary CPM-dispersions of $R_{0,0}$ and $X_{gc,cn}(s,t)$, respectively. Then, the CPM-dispersion of $R_{1,gc,cn}(m,n,s,t)$ gives the following $(mt + s) \times nt$ array $H_{1,gc,cn,qc}(q - 1, q - 1)$ of binary CPMs and ZMs of size $(q - 1) \times (q - 1)$:

$$H_{1,gc,cn,qc}(q - 1, q - 1) = \begin{bmatrix}
CPM(R_{0,0}) & & & \\
& CPM(R_{0,0}) & & \\
& & \ddots & \\
& & & CPM(X_{gc,cn}(s,t))
\end{bmatrix}_{(mt + s) \times nt}(q - 1)$$

(6)

The array $H_{1,gc,cn,qc}(q - 1, q - 1)$ is a binary $(mt + s) \times nt(q - 1)$ matrix. Since the base matrix $R_{1,gc,cn}(m,n,s,t)$ satisfies the $2 \times 2$ SM-constraint, $H_{1,gc,cn,qc}(q - 1, q - 1)$ satisfies the RC-constraint. The null space of $H_{1,gc,cn,qc}(q - 1, q - 1)$ gives a CN-based QC-GC-LDPC code of length $nt(q - 1)$, denoted by $C_{1,gc,cn,qc}$. The Tanner graph $\Gamma_{1,gc,cn,qc}(q - 1, q - 1)$ is an expansion of the graph $\Gamma_{1,gc,cn}$ by a factor of $q - 1$. From the graphical point of view, $\Gamma_{1,gc,cn,qc}(q - 1, q - 1)$ is obtained by taking $q - 1$ copies of $\Gamma_{1,gc,cn}$ and then connecting the copies by permuting the edges among the copies. Hence, $\Gamma_{1,gc,cn,qc}$ may be regarded as a protograph and the code $C_{1,gc,cn,qc}$ can be regarded as a protograph-based LDPC code [10].

In forming the array $R_{1,gc,cn}(m,n,s,t)$, we can use $t$ different member matrices in the set $\{R_{0,0}, R_{0,1}, \ldots, R_{0,r-1}\}$ as the matrices on the main diagonal of the upper $t \times t$ subarray of $R_{1,gc,cn}(m,n,s,t)$. In this case, we obtain the following base matrix:

$$R_{1,gc,cn}^+(m,n,s,t) = \begin{bmatrix}
R_{0,0} & & & \\
& R_{0,1} & & \\
& & \ddots & \\
& & & R_{0,t-1}
\end{bmatrix}_{mt \times t}(q - 1)$$

(7)

The CPM-dispersion of $R_{1,gc,cn}^+(m,n,s,t)$ gives an $(mt + s) \times nt$ array $H_{1,gc,cn,qc}(q - 1, q - 1)$ whose null space gives a time varying CN-based QC-GC-LDPC code.

For different choices of parameters, $r, l, m, n, s$ and $t$, we can construct a large family of CN-based QC-GC-LDPC codes from a given field $GF(q)$ with various lengths and rates.

In the following example, we construct three high-rate codes based on the same finite field using three different sets of parameters, $m, n, s$ and $t$. The codes have very good error performance, especially the error-floor performance. We label these three codes with indices 0, 1 and 2, respectively.

**Example 1.** Consider the prime field $GF(127)$ for the code construction. First, we construct a $126 \times 126$ cyclic mother matrix $B_1$ over $GF(127)$ in the form given by (3) which satisfies the $2 \times 2$ SM-constraint.

Factor $127 - 1 = r \times l = 6 \times 21$ and set $r = 6$ and $l = 21$. Then, the matrix $B_1$ can be viewed as a $6 \times 6$ array of $21 \times 21$ submatrices, whose first row-block consists of six $21 \times 21$ submatrices $W_{0,0}, W_{0,1}, \ldots, W_{0,5}$. Taking six $3 \times 21$ submatrices, $R_{0,0}, R_{0,1}, \ldots, R_{0,5}$ from these submatrices $W_{0,0}, W_{0,1}, \ldots, W_{0,5}$ (i.e., choose $m = 3, n = 21$), we obtain a $6 \times 6$ array $R_1(3,21)$ of $21 \times 21$ submatrices over $GF(127)$ in the form given by (4). The submatrix $R_{0,0}$ is a $3 \times 21$ matrix over $GF(127)$. The set $\Pi^*$ consists of rows in $R_0, R_0, \ldots, R_0$. Choose $t = 6$ and $s = 1$. Then, the base matrix $R_{1,gc,cn}(3,21,1,6)$ consists of 6 copies of $R_{0,0}$ lying on its main diagonal and 1 row (or $1 \times 6$ array of $1 \times 21$ submatrices) from the set $\Pi^*$ (the row from $\Pi^*$ is chosen at random). $R_{1,gc,cn}(3,21,1,6)$ is a $19 \times 126$ matrix over $GF(127)$ and satisfies the $2 \times 2$ SM-constraint. The binary CPM-dispersion of $R_{1,gc,cn}(3,21,1,6)$ gives a $19 \times 126$ array $H_{1,gc,cn,qc}(126,126)$ of CPMs and ZMs of size $126 \times 126$. The array $H_{1,gc,cn,qc}(126,126)$ is a $2394 \times 15876$ binary matrix and has two column weights, 3 and 4, two row weights, 21 and 125, and average column and row weights, 3.99 and 26.47, respectively. The null space of $H_{1,gc,cn,qc}(126,126)$ gives a $(15876, 13494)$ CN-based QC-GC-LDPC code $C_{1,gc,cn,qc,0}$ with rate 0.85. The Tanner graph $\Gamma_{1,gc,cn,qc,0}(126,126)$ of the code $C_{1,gc,cn,qc,0}$ has girth 6 and contains 204,876 cycles of length 6 and 21,677,544 cycles of length 8. The bit error rate (BER) and block error rate (BLER) performances of the code over the AWGN channel are shown in Fig. 1(a).

Suppose we choose another two sets of parameters, $r = 3, l = 42, m = 2, n = 42, s = 2, t = 3$ and $r = 3, l = 42, m = 1, n = 42, s = 3, t = 3$. Based on these two sets of parameters and the construction method described above, we can construct another two $QC$-GC-LDPC codes, $C_{1,gc,cn,qc,1}$ and $C_{1,gc,cn,qc,2}$. $C_{1,gc,cn,qc,1}$ is a $(15876, 14871)$ CN-based QC-GC-LDPC code with rate 0.9367. The average column and row weights of $C_{1,gc,cn,qc,1}$ are 3.9841 and 62.75, respectively. The Tanner graph of $C_{1,gc,cn,qc,1}$ has girth 6 and contains 1,675,296 cycles of length 6 and 384,784,778 cycles of length 8. $C_{1,gc,cn,qc,2}$ is a $(15876, 15120)$ CN-based QC-GC-LDPC code with rate 0.9524. The average column and row weights of $C_{1,gc,cn,qc,2}$ are 3.9762 and 83.5, respectively. The Tanner graph of $C_{1,gc,cn,qc,2}$ has girth 6 and contains 3,783,780 cycles of length 6 and 1,037,792,362 cycles of length 8. The BER and BLER performances of these two codes $C_{1,gc,cn,qc,1}$ and $C_{1,gc,cn,qc,2}$ decoded with 50 iterations of the MSA over the AWGN channel are included in Fig. 1(a).

The three high-rate codes perform well. Consider the $(15876, 14871)$ QC-GC-LDPC code $C_{1,gc,cn,qc,1}$ with rate 0.9367. From Fig. 1(a), we see that the code has no visible error-floor all the way down to a BER of $10^{-10}$. At a BER of
Note that the CPM-dispersion $\text{CPM}(R_{0,0})$ of $R_{0,0}$ in the base matrix $R_{1,gc,cn}(m, n, s, t)$ is an RC-constrained $m \times n$ array of CPMs and/or ZMs of size $(q-1) \times (q-1)$. Its null space gives a QC-LDPC code $C_{0,0}$ of length $n(q-1)$. From the structure of the parity-check matrix array $H_{1,gc,cn,qe}(q-1, q-1)$ of the CN-based QC-GC-LDPC code $C_{1,gc,cn,qc}$ displayed in (6), we readily see that each codeword $v = (v_0, v_1, \ldots, v_{t-1})$ of $C_{1,gc,cn,qc}$ consists of $t$ sections, each containing $n(q-1)$ code symbols. Each section $v_i$ of $v$ is a codeword of the code $C_{0,0}$ and the cascaded $t$ sections of $v$ in the order $v_0, v_1, \ldots, v_{t-1}$ must satisfy the $s(q-1)$ parity-check constraints specified by the lower subarray CPM($X_{gc,cn}(s, t)$) of $H_{1,gc,cn,qe}(q-1, q-1)$. From the classical coding point of view, the code $C_{1,gc,cn,qc}$ may be regarded as a cascaded LDPC code. The cascading structure of $C_{1,gc,cn,qc}$ may be used for correcting errors that are distributed among the $t$ sections of a received codeword. If the error pattern in each section of the received codeword is a correctable error pattern for the code $C_{0,0}$ with a specific iterative decoding algorithm, the decoding will be successful and correct. Then, the $t$ successfully decoded sections give the transmitted codeword.

Based on the structure described above, a local/global two-phase iterative decoding scheme can be devised. Suppose a codeword $v = (v_0, v_1, \ldots, v_{t-1})$ is transmitted. Let $r = (r_0, r_1, \ldots, r_{t-1})$ be the received word. The decoding starts with a local phase of decoding. Each received section $r_i$ is decoded using a local iterative decoder based on the local code $C_{0,0}$ with a fixed number of iterations. If all the sections are successfully decoded, the $t$ decoded sections form a locally decoded codeword $v^* = (v_0^*, v_1^*, \ldots, v_{t-1}^*)$. Then, we check whether this locally decoded codeword satisfies the $s(q-1)$ parity-check constraints imposed by the $s(q-1)$ global CNs. If it does, the locally decoded codeword is a codeword in the CN-based QC-GC-LDPC code $C_{1,gc,cn,qc}$ and it is delivered to the user. If the locally decoded codeword $v^*$ does not satisfy the $s(q-1)$ global CN-constraints, a global decoder based on the entire code $C_{1,gc,cn,qc}$ is activated to decode the received vector $r$ with symbol reliability measures provided by the local decoders and the channel information as inputs to perform the global phase of decoding.

There are two simple approaches to switch the decoding from the local phase to the global phase. The first one is to complete the local decoding of all the received sections, and then the decoded information (LLRs) of all the decoded sections (successfully and unsuccessfully decoded) and the channel information are passed to the global decoder to process the received codeword $r$. The other approach is to switch to the global phase decoding as soon as a local decoder fails to decode a received section. Then, the global decoder processes the received vector $r$ either with only the channel information as inputs or the combined channel information with information of successfully decoded sections as inputs.

Basicly, the global phase of decoding is needed only when
the distribution of errors is bad or the number of errors is large. Intuitively, the average number of computations should be lower than that of using only a global decoder. In the decoder implementation, we can use one local decoder based on the local code $C_{0,0}$ to process all the received sections sequentially, or use a group of ($t$ or less) identical local decoders to process a group of ($t$ or less) consecutive received sections in parallel. Of course, pipeline processing is possible. Clearly, using only one local decoder results in the lowest decoder complexity, but requires the longest decoding time. Fully parallel local phase decoding using $t$ local decoders gives the shortest decoding time, but requires highest decoder complexity. Partially parallel local phase decoding using a group of local decoders (less than $t$) gives a trade-off between decoding complexity and decoding time. Architecture design of such a local/global decoder would be an interesting problem for further investigation.

V. CONSTRUCTION OF CN-BASED QC-GC-LDPC CODES FOR CORRECTING BURSTS OF ERASURES

In this section, we present a method for constructing CN-based QC-GC-LDPC codes which are capable of correcting erasures clustered in bursts, called erasure-bursts. The construction is based on partitioning a submatrix of a $2 \times 2$ SM-constrained base matrix $B$ into an array of square matrices and splitting each square matrix into two triangular matrices. In the code construction, we use the base matrix $B_2$ given by (2). Note that we can also use $B_1$ as the base matrix for code construction. The purpose of using $B_2$ rather than $B_1$ is to show that the algebraic method for constructing LDPC codes is very flexible.

Let $\ell$ and $t$ be two positive integers such that $4\ell t < p$. Suppose we remove the last $p - 4\ell t$ columns from $B_2$. We obtain the following $p \times 4\ell t$ subarray $B_2(\ell, t)$ of $B_2$:

$$B_2(\ell, t) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \beta & \beta^2 & \cdots & \beta^{4\ell t - 1} \\ 1 & \beta^2 & (\beta^2)^2 & \cdots & (\beta^2)^{4\ell t - 1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \beta^{p-1} & (\beta^{p-1})^2 & \cdots & (\beta^{p-1})^{4\ell t - 1} \end{bmatrix}.$$  \hfill (8)

Since $B_2(\ell, t)$ is a submatrix of $B_2$, it satisfies the $2 \times 2$ SM-constraint. Take the first $2t$ rows from $B_2(\ell, t)$ and divide each row into $4\ell$ sections, each consisting of $t$ components. From these $2t$ sectionalized rows, we form the following $2 \times 4\ell t$ array $W(2, 4\ell)$ of $t \times t$ matrices over $\mathbb{F}(q)$:

$$W(2, 4\ell) = \begin{bmatrix} W_0 & W_{1,0} & \cdots & W_{0,4\ell t-1} \\ W_1 & W_{1,1} & \cdots & W_{1,4\ell t-1} \end{bmatrix}. \hfill (9)$$

For $0 \leq i < 2$ and $0 \leq j < 4\ell$, we express the $t \times t$ constituent matrix $W_{i,j}$ of $W(2, 4\ell)$ in terms of its NB entries as follows:

$$W_{i,j} = \begin{bmatrix} w_{i,j,0,0} & w_{i,j,0,1} & \cdots & w_{i,j,0,t-1} \\ w_{i,j,1,0} & w_{i,j,1,1} & \cdots & w_{i,j,1,t-1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i,j,t-1,0} & w_{i,j,t-1,1} & \cdots & w_{i,j,t-1,t-1} \end{bmatrix}. \hfill (10)$$

Now, we cut $W_{i,j}$ along its main diagonal and form two triangular submatrices. There are two different types of cutting, type-1 and type-2. The type-1 cutting is shown as follows:

$$W^1_{i,j,low} = \begin{bmatrix} w_{i,j,0,0} & 0 & \cdots & 0 \\ w_{i,j,1,0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ w_{i,j,t-1,0} & w_{i,j,t-1,1} & \cdots & w_{i,j,t-1,t-1} \end{bmatrix}, \hfill (11)$$

$$W^1_{i,j,up} = \begin{bmatrix} 0 & w_{i,j,0,1} & \cdots & w_{i,j,0,t-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \hfill (12)$$

The two matrices $W^1_{i,j,low}$ and $W^1_{i,j,up}$ are called type-1 lower and upper triangular matrices of $W_{i,j}$, respectively. Note that the entries on and below the main diagonal of the type-1 lower triangular matrix $W^1_{i,j,low}$ of $W_{i,j}$ are all nonzero and the entries above the main diagonal are all zeros. The column weights of $W^1_{i,j,low}$ range from 1 to $t$.

Type-2 cutting of $W_{i,j}$ is shown as below:

$$W^2_{i,j,low} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ w_{i,j,1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ w_{i,j,t-1,0} & w_{i,j,t-1,1} & \cdots & 0 \end{bmatrix}, \hfill (13)$$

$$W^2_{i,j,up} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ w_{i,j,0,1} & w_{i,j,0,2} & \cdots & w_{i,j,0,t-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \hfill (14)$$

The two matrices $W^2_{i,j,low}$ and $W^2_{i,j,up}$ are called type-2 lower and upper triangular matrices of $W_{i,j}$, respectively. With the type-2 cutting, the entries on and above the main diagonal of the upper triangular matrix $W^2_{i,j,up}$ are nonzero and entries below the main diagonal are all zeros. Now, we divide the first row-block $W_0$ of $W(2, 4\ell)$ given by (9) into $\ell$ sections, each consisting of 4 consecutive constituent matrices $W_{0,4k}$, $W_{0,4k+1}$, $W_{0,4k+2}$, $W_{0,4k+3}$ for $0 \leq k < \ell$, each being a $t \times t$ matrix. We replace the four constituent matrices $W_{0,4k}$, $W_{0,4k+1}$, $W_{0,4k+2}$, $W_{0,4k+3}$ by $W_{1,4k,low}$, $O$, $W_{2,4k+2,up}$, $O$, respectively, where $O$ is a $t \times t$ zero matrix. Next, we divide the second row-block $W_1$ of $W(2, 4\ell)$ into $\ell$ sections, each consisting of 4 consecutive constituent matrices $W_{1,4k}$, $W_{1,4k+1}$, $W_{1,4k+2}$, $W_{1,4k+3}$ for $0 \leq k < \ell$. The latter problem will be considered in a future work.
\( \ell \), each being a \( t \times t \) matrix. Now we replace the four constituent matrices \( W_{1,4k}^1, W_{1,4k+1}, W_{1,4k+2}, W_{1,4k+3} \) by \( O, W_{1,4k+1,low}^1, O, W_{1,4k+3,up}^1 \), respectively. By performing the above replacements, we form a \( 2 \times 4t \) array \( W^*(2, 4\ell) \) in the form of (15) on the top of the next page of triangular and zero matrices of size \( t \times t \).

The array \( W^*(2, 4\ell) \) is a \( 2t \times 4t \ell \) matrix. Label the rows and columns of \( W^*(2, 4\ell) \) from 0 to \( 2t-1 \) and 0 to \( 4t\ell-1 \), respectively. \( W^*(2, 4\ell) \) has the following properties: 1) Its two row-blocks are disjoint, i.e., a row in the first row-block and a row in the second row-block do not have any position where they both have nonzero components; 2) The compositions of submatrices in the two row-blocks are identical; 3) Each row in either row-block of \( W^*(2, 4\ell) \) has \( 2t \) zero-spans (including end around zero-spans), each consisting of \( 2t-1 \) consecutive zeros which are confined by two nonzero elements; 4) In each column, there is a single nonzero entry in a row followed by exactly \( 2t-1 \) zeros; 5) The column weights of \( W^*(2, 4\ell) \) are from 1 to \( t \) with an average of \((t + 1)/2\); and 6) \( W^*(2, 4\ell) \) has constant row weight \((t + 1)\ell\). Since two row-blocks of \( W^*(2, 4\ell) \) are disjoint and their compositions of submatrices are identical, their associated Tanner graphs are structurally identical (different only in edge labels) and disjoint. Each zero-span in \( W^*(2, 4\ell) \) is said to have length \( 2t-1 \).

Let \( f \) be a positive integer such that \( f \leq p - 2t \). Next, we take \( f \) rows from the last \( p-2t \) rows of \( B_2(\ell, t) \) below the \( 2t \) rows of the \( 2t \times 4t \ell \) matrix \( W(2, 4\ell) \) to form an \( f \times 4t \ell \) matrix, denoted by \( W_{ge,cn}(f, 4t\ell) \), which has constant column weight \( f \) and row weight \( 4t \ell \). Therefore, \( H_{2,ge,cn,qc}(q - 1, q - 1) \) has multiple column weights, ranging from \( f + 1 \) to \( f + t \) and two row weights \((t + 1)\ell \) and \( 4t \ell \). Its average column weight is \( f + (t + 1)/2 \).

Since \( B_{2,ge,cn}(2t + f, 4t\ell) \) satisfies the \( 2 \times 2 \) SM-constraint, \( H_{2,ge,cn,qc}(q - 1, q - 1) \) satisfies the RC-constraint. It follows from the structure of \( W^*(2, 4\ell) \) that the upper subarray CPM(\( W^*(2, 4\ell) \)) of \( H_{2,ge,cn,qc}(q - 1, q - 1) \) consists of \( 2t \) type-1 lower triangular arrays of CPMs and ZMs of size \((q - 1) \times (q - 1), 2t \) type-2 upper triangular arrays of CPMs and ZMs of size \((q - 1) \times (q - 1), 4t \) ZMs of size \((q - 1) \times (q - 1) \). It follows from the zero-span structure of \( W^*(2, 4\ell) \) that each row in CPM(\( W^*(2, 4\ell) \)) contains \( 2t \) zero-spans, each of length at least \((2t-1)(q-1)\). In each column of \( H_{2,ge,cn,qc}(q - 1, q - 1) \), there is exactly one 1-entry in a row of CPM(\( W^*(2, 4\ell) \)) followed by a span of at least \((2t-1)(q-1)\) zeros. The null space of \( H_{2,ge,cn,qc}(q - 1, q - 1) \) gives an irregular CN-based QC-GC-LDPC code \( C_{2,ge,cn,qc} \).

On the BEC, if erasures are confined to \( b \) consecutive positions, the first and last of which are erasures, we call such an erasure pattern, denoted by \( e_{EB} \), an erasure-burst of length \( b \). The subscript “EB” in \( e_{EB} \) stands for “erasure-burst”. Next, we show that any erasure-burst \( e_{EB} \) of length less than \((2t-1)(q-1)+1\) can be corrected (or resolved) by the code \( C_{2,ge,cn,qc} \) constructed above. To this end, we need to examine the structure of the upper subarray CPM(\( W^*(2, 4\ell) \)) of \( H_{2,ge,cn,qc}(q - 1, q - 1) \). CPM(\( W^*(2, 4\ell) \)) consists of \( \ell \) sections. Each section of CPM(\( W^*(2, 4\ell) \)), denoted by \( H_{2,k} \), with \( 0 \leq k \leq \ell \), is a \( 2 \times 4 \) array in the form of (18) on the top of next page.

The matrices CPM(\( W_{1,4k,low}^1 \)) and CPM(\( W_{1,4k+1,low}^1 \)) in \( H_{2,k}(2, 4) \) are two type-1 \( t \times t \) lower triangular arrays of CPMs and ZMs of size \((q - 1) \times (q - 1), 1 \), CPM(\( W_{2,4k+2,up}^2 \)) and CPM(\( W_{2,4k+3,up}^2 \)) are two type-2 upper triangular arrays of CPMs and ZMs of size \((q - 1) \times (q - 1), 0 \), and CPM(\( O \)) is a \( t \times t \) array of ZMs of size \((q - 1) \times (q - 1), 0 \). Each section \( H_{2,k}(2, 4) \) of \( H_{2,ge,cn,qc}(q - 1, q - 1) \) consists of 4 column-blocks, each consisting of a type-1 \( t \times t \) lower (or a type-2 \( t \times t \) upper) triangular array and a \( t(q - 1) \times t(q - 1) \) ZM. Each column-block of \( H_{2,k}(2, 4) \) consists of \( t \) columns of CPMs and ZMs of size \((q - 1) \times (q - 1) \). Note that each collection of 4 consecutive sections of CPM(\( W^*(2, 4\ell) \)), including end around case, has similar structure.

Suppose an erasure-burst \( e_{EB,0} \) of length \( b \), less than or equal to \((2t-1)(q-1)+1\), occurs during transmission of a codeword in \( C_{2,ge,cn,qc} \). It follows from the zero-span and triangular structure of the upper subarray CPM(\( W^*(2, 4\ell) \)) of \( H_{2,ge,cn,qc}(q - 1, q - 1) \) that there is one row in CPM(\( W^*(2, 4\ell) \)) that checks one and only one erasure in \( e_{EB,0} \). Based on the zero parity-check constraint formed by this row, we can recover the erased code symbol at the location of the erasure being checked. As a result, we reduce the number of erasures in \( e_{EB,0} \) by one and obtain a new erasure-burst \( e_{EB,1} \). Again, there is one row in CPM(\( W^*(2, 4\ell) \)) that checks one and only one erasure in \( e_{EB,1} \). Based on this row, we can recover the erased code symbol at the location of the
erasure in $e_{EB,1}$ which is checked. This reduces the number of erasures in $e_{EB,1}$ by 1. We can continue this erasure-recovering process until all the erased code symbols at the locations of the erasures in $e_{EB,0}$ are recovered.

Scattered random error and/or erasures may also be recovered by the entire parity-check matrix $H_{2,gc,cn,qc}$ of the code $C_{2,gc,cn,qc}$.

**Example 2.** Let $GF(2^7)$ be the field for the code construction. Since $2^7 - 1 = 127$ is a prime, the largest prime factor $p$ is 127, i.e., the order of $\beta$ is 127. We can construct a $127 \times 127$ base matrix $B_2$ in the form of (2). Choose $t = 5$, $\ell = 6$ and $f = 2$. Using the above construction process, we can construct the following $1524 \times 15240$ parity-check matrix:

$$H_{2,gc,cn,qc}(127,127) = \begin{bmatrix} CP(M^{*}(2,24)) \\ CP(M_{gc,cn}(2,120)) \end{bmatrix},$$

which is a $12 \times 120$ array of CPMs and ZMs of size $127 \times 127$. The upper submatrix $CP(M^{*}(2,24))$ of $H_{2,gc,cn,qc}(127,127)$ consists of 6 sections. Each section is a $2 \times 4$ array which consists of two $5 \times 5$ type-1 lower triangular arrays of CPMs and ZMs of size $127 \times 127$, two $5 \times 5$ type-2 upper triangular arrays of CPMs and ZMs of size $127 \times 127$ and 4 ZMs of size $635 \times 635$ in the form of (18). Every row of $CP(M^{*}(2,24))$ consists of 12 zero-spans, each of length at least 1143. The lower submatrix $CP(M_{gc,cn}(2,120))$ of $H_{2,gc,cn,qc}(127,127)$ is a $2 \times 120$ array of CPMs of size $127 \times 127$.

The null space of $H_{2,gc,cn,qc}(127,127)$ gives a $(15240, 13771)$ CN-based QC-GC-LDPC code $C_{2,gc,cn,qc}$ with rate 0.9901, which is capable of correcting any erasure-burst of length at least up to 1144. The associated Tanner graph has girth 6 and contains 2,987,167 cycles of length 6 and 660,098,629 cycles of length 8. Its BER and BLER performances over the AWGNC decoded with 50 iterations of the MSA are shown in Fig. 2(a) and UEBLR and UEBR performances over the BEC are shown in Fig. 2(b). On the AWGNC, at a BER of $10^{-7}$, the code performs 1.0 dB away from the Shannon limit and 0.75 dB away from the SPB at a BLER of $10^{-4}$. On the BEC, at a UEBR of $10^{-8}$, it performs 0.04 away from the Shannon limit. We see that the code performs well on both the AWGNC and the BEC, in addition to its large erasure-burst correction capability.

The method presented above can be generalized to construct a CN-based QC-GC-LDPC code capable of correcting longer erasure-bursts. For example, suppose we chose $t$ and $\ell$ such that $6\ell t < p$. With these parameters, we construct a $3 \times 6\ell$
base array $W^*(3, 6\ell)$ of triangular and zero matrices of size $t \times t$ which consists of $\ell$ sections. Each section $W_k^*(3, 6)$ with $0 \leq k < \ell$ of $W^*(3, 6\ell)$ is a $3 \times 6$ array of type-1 lower triangular, type-2 upper triangular and zero matrices of size $t \times t$ in the form of (19) on the top of next page.

Then

$$W^*(3, 6\ell) = [W_0^*(3, 6) \quad W_1^*(3, 6) \quad \cdots \quad W_{\ell-1}^*(3, 6)].$$ (20)

Each row of $W^*(3, 6\ell)$ consists of $2\ell$ zero-spans, each of length $3t-1$. The CPM-dispersion of $W^*(3, 6\ell)$ gives an array CPM($W^*(3, 6\ell)$) in which each row has $2\ell$ zero-spans, each of length at least $(3t-1)(q-1)$. Using CPM($W^*(3, 6\ell)$) and CPM($W_{gc,cn}(f, 6\ell t)$) to replace CPM($W^*(2, 4\ell)$) and CPM($W_{gc,cn}(f, 4\ell t)$) in (17), respectively, we obtain a parity-check matrix $H^*_{2,gc,cn,qc}(q-1, q-1)$ as follows:

$$H^*_{2,gc,cn,qc}(q-1, q-1) = \begin{bmatrix} \text{CPM}(W^*(3, 6\ell)) \\ \text{CPM}(W_{gc,cn}(f, 6\ell t)) \end{bmatrix}.$$ (21)

The null space of $H^*_{2,gc,cn,qc}(q-1, q-1)$ gives a CN-based QC-GC-LDPC code $C^*_{2,gc,cn,qc}$ which is capable of correcting any erasure-burst of length at least up to $(3t-1)(q-1)+1$.

The above construction can be further generalized to construct a CN-based QC-GC-LDPC code to correct erasure-bursts of lengths at least up to $(ct-1)(q-1)+1$ with $2c\ell t < p$ and $c \geq 2$.

**Example 3.** This example is a continuation of Example 2. In this example, we set $t = 5$, $\ell = 4$ and $f = 2$. Based on this new set of parameters, we construct the following parity-check array $H^*_{2,gc,cn,qc}(127, 127)$ of CPMs and ZMs of size $127 \times 127$:

$$H^*_{2,gc,cn,qc}(127, 127) = \begin{bmatrix} \text{CPM}(W^*(3, 24)) \\ \text{CPM}(W_{gc,cn}(2, 120)) \end{bmatrix}.$$ (22)

Every row of the upper subarray CPM($W^*(3, 24)$) of $H^*_{2,gc,cn,qc}(127, 127)$ consists of 8 zero-spans, each of length at least 1778. The null space of $H^*_{2,gc,cn,qc}(127, 127)$ gives a 15240,13082 CN-based QC-GC-LDPC code $C^*_{2,gc,cn,qc}$ with rate 0.8584. The code is capable of correcting any erasure-burst of length at least up to 1789. Its Tanner graph has girth 6 and contains 1,611,376 cycles of length 6 and 334,328,389 cycles of length 8.

The BER and BLER performances of this code over the AWGNC decoded with 50 iterations of the MSA are shown in Fig. 3(a) and its UEBLR and UEBR performances over the BEC are shown in Fig. 3(b). On the AWGNC, at a BER of $10^{-5}$, the code performs within 1.2 dB away from the Shannon limit and 0.9 dB away from the SPB at a BLER of $10^{-5}$. On the BEC, at a UEBR of $10^{-7}$, it performs 0.04 away from the Shannon limit. We see that the code performs well on both the BEC and the AWGNC, in addition to its large erasure-burst correction capability.

**VI. CONCLUSION AND REMARKS**

In this paper, we presented two algebraic methods for constructing CN-based quasi-cyclic globally coupled LDPC codes. Both construction methods are based on dispersing (or expanding) the entries of a $2 \times 2$ SM-constrained base matrix over a nonbinary finite field into binary CPMs and/or ZMs of the same size. Two classes of $2 \times 2$ SM-constrained base matrices were presented. Examples showed that codes constructed perform well on both the AWGNC and the BEC. Codes constructed by one method are effective for correcting erasure bursts clustered in bursts. Also presented in this paper was a two-phase local/global iterative scheme for decoding the CN-based GC-LDPC codes. The decoding scheme allows correction of local random and global errors and/or erasures.
\[ W_k^{(3,6)}(3) = \begin{bmatrix} W_{0,k,low}^1 & 0 & 0 & W_{0,k+1,up}^2 & 0 & 0 \\ 0 & W_{1,k,low}^1 & 0 & 0 & W_{1,k+1,up}^2 & 0 \\ 0 & 0 & W_{2,k,low}^1 & 0 & 0 & W_{2,k+1,up}^2 \end{bmatrix}. \] (19)

The parity-check matrix of the direct product of two LDPC codes can be put into the form given by (6) with the copies of the parity-check matrix of one code lying on the main diagonal of the top subarray and copies of the rows of the parity-check matrix of the other code forming the global CNs [4], [9]. Any matrix over a nonbinary field besides the matrices given by (1) and (2) that satisfies the \( 2 \times 2 \) SM-constraint can be used as the mother base matrix to construct CN-based quasi-cyclic globally coupled LDPC codes. The cyclic base matrix given by (3) can also be used to construct spatially coupled LDPC codes.

REFERENCES


