A concentration result for stochastic approximation
(Extended abstract)

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Abstract—This is a summary of the main results of [9] concerning concentration of interpolated iterates of a Robbins-Monro scheme around the trajectory of a limiting differential equation from some time on.

I. INTRODUCTION

The Robbins-Monro stochastic approximation scheme [8], originally proposed for finding roots of a nonlinear function given its noisy measurements, is one of the major workhorses of statistical computation, signal processing, machine learning, etc. The basic scheme is the $d$-dimensional iteration

$$
x(n+1) = x(n) + a(n)[h(x(n)) + M(n+1)], \ n \geq 0.
$$  

(1)

The term in square brackets is the noisy measurement of $h(x(n))$. The noise $\{M(n)\}$ is assumed to be a martingale difference sequence, i.e.,

$$
E[M(n+1)|F_n] = 0
$$

where $F_n := \sigma(x_0, M(m), m \leq n)$ for $n \geq 0$. The key innovation of Robbins and Monro was their clever choice of the stepsize schedule $a(n) > 0$. They imposed the conditions

$$
\sum_{n} a(n) = \infty, \quad \sum_{n} a(n)^2 < \infty.
$$  

(2)

That is, $a(n)$ is slowly decreasing in a precise sense. These conditions turn out to be just right to suppress the discretization errors and noise asymptotically.

One of the more successful approaches to the analysis of the asymptotic behaviour of (1) has been the so called ‘o.d.e.’ (for ‘ordinary differential equation’) method due to Dorevitskii and Fradkov [4]. This approach views (1) as a noisy Euler scheme for the o.d.e.

$$
\dot{x}(t) = h(x(t)), \ t \geq 0.
$$  

(3)

Under suitable conditions, it can be shown that (1) asymptotically tracks the asymptotic behaviour of (3). See [3], Chapter 2 for a modern exposition.

This analysis is based on a key lemma which is stated as follows. Define ‘algorithmic time’ $\{t(n)\}$ by: $t(0) = 0, t(n) = \sum_{m=0}^{n-1} a(m), n \geq 0$. Define $\bar{x}(t), t \geq 0$, by: $\bar{x}(t(n)) = x(n), \ n \geq 0$, with linear interpolation on each interval $[t(n), t(n+1)]$. This renders $\bar{x}(\cdot)$ piecewise linear and continuous. For $s \geq 0$, let $x^*(t), t \geq s$, denote the trajectory of (3) on $[s, \infty)$ with $x^*(s) = \bar{x}(s)$. Then for any $T > 0$,

$$
\lim_{s \to \infty} \max_{t \in [s, s+T]} \| \bar{x}(s + t) - x^*(t) \| = 0
$$

(4)
a.s. This follows essentially by Gronwall inequality, with a suitable martingale convergence theorem ensuring the asymptotic vanishing of the noise contribution, while $a(n) \to 0$ does the same for discretization error.

Our aim here is to go a step further and establish bounds for the probability that $\bar{x}(\cdot)$ remains in a tubular neighbourhood of a trajectory of (3) from some time on. There have been results of this flavour earlier, see, e.g., Chapter 4 of [3] and [6]. (See also [5] for some related work.) The present work significantly improves upon these. We summarize the main highlights in the next section. The full details can be found in [9]. Section 3 concludes with a brief discussion of the results.

II. MAIN RESULT

We shall assume that $h : \mathcal{R}^d \to \mathcal{R}^d$ is twice continuously differentiable and $\{M(n)\}$ satisfies the conditional tail bound:

$$(\dagger) \quad \text{There exist continuous functions } f, g : \mathcal{R}^d \to (0, \infty) \text{ such that}
$$

$$
P(\|M(n+1)\| > u|F_n) \leq f(x(n)) e^{-g(x(n))u}, \ n \geq 0.
$$  

(5)

Let $x^*$ be an asymptotically stable equilibrium of (3) with domain of attraction $A$ and $B \subset A$ a prescribed set satisfying hypotheses that we shall soon specify. By the converse Liapunov theorem, there exists a smooth Liapunov function $V : A \to [0, \infty)$ with $\lim_{x \in \mathcal{A}, x \to \partial A} V(x) = \infty$ and

$$
\langle \nabla V(x), h(x) \rangle < 0 \ \forall \ x \neq x^*.
$$

Define

$$
V^z := \{ x \in A : V(x) \leq z \}, \ z > 0,
$$

$$
\mathcal{N}_{\epsilon_0}(D) := \{ x \in A : \min_{y \in D} \| y - x \| \leq \epsilon_0 \}, \ D \subset A.
$$

We assume that there exist $0 < \epsilon \leq \epsilon_0, r \geq r_0 > 0$ such that

$$
\{ x \in A : \| x - x^* \| \leq \epsilon \} \subset B \subset V^{r_0} \subset \mathcal{N}_{\epsilon_0}(V^{r_0}) \subset V^r \subset A.
$$
Since $x^*$ is asymptotically stable, the Jacobian matrix $Dh(x^*)$ of $h$ at $x^*$ is stable, i.e., it has all eigenvalues $\{\lambda_i\}$ is the open left half plane. Let

$$\lambda_{\text{min}} := \min_i \{-\Re(\lambda_i)\} > 0$$

and fix $\lambda' \in (0, \lambda_{\text{min}})$, $\kappa \in (0, 1)$. Then for a suitable $K \geq 1$,

$$\left\| e^{Dh(x^*) t} \right\| \leq K e^{-\lambda' t}, \quad t \geq 0.$$  

Define

$$\lambda := \left(1 - \frac{\kappa}{K^2}\right) \lambda' < \lambda'$$

and for some $n_0 \geq 0$,

$$\beta(n) := \max_{n_0 \leq k \leq n \leq 1} \left( e^{-\lambda \sum_{i=k+1}^{n-1} a_i} a_k \right).$$

Our main result can then be stated as follows.

**Theorem** For sufficiently large $n_0 \geq 0$ and $T > 0$, $\{x(n)\}$ satisfy the following:

1. For $\epsilon \leq 1$,

$$P \left( \|\bar{x}(t) - x^*\| \leq \epsilon \forall t \geq t(n_0) + T + 1|\bar{x}(t(n_0)) \in B \right) 
\geq 1 - \sum_{n \geq n_0} C_1 e^{-C_2 \epsilon^2 \frac{T}{n^{\frac{1}{3}}} } - \sum_{n \geq n_0} C_2 e^{-C_2 \epsilon^2 \frac{T}{n^{\frac{1}{3}}} }.$$  

2. For $\epsilon > 1$,

$$P \left( \|\bar{x}(t) - x^*\| \leq \epsilon \forall t \geq t(n_0) + T + 1|\bar{x}(t(n_0)) \in B \right) 
\geq 1 - \sum_{n \geq n_0} C_1 e^{-C_2 \epsilon^2 \frac{T}{n^{\frac{1}{3}}} } - \sum_{n \geq n_0} C_2 e^{-C_2 \epsilon^2 \frac{T}{n^{\frac{1}{3}}} }.$$  

Here $C_1, C_2$ are constants depending only on $\lambda, d, r$.

See [9] for a detailed proof. Here we highlight two salient features.

1) We treat (1) as a regular perturbation of the o.d.e. (3) and use Alexeev’s nonlinear variation of constants formula which allows us to write the difference between perturbed and unperturbed trajectories in terms of the linearized o.d.e. We take the latter to be the constant trajectory $x(t) \equiv x^*$. The exponential stability of the linearized o.d.e. gives us the additional handle that helps us improve significantly upon existing results.

2) Another key component is a variation of a martingale concentration inequality of Liu and Watbled [7] for martingales satisfying an exponentially decaying conditional tail probability condition (5). We establish this as a part of our work.

## III. Discussion

Some of the highlights of this result are as follows.

1) This is a local result, giving a bound on the so-called lock-in probability [2] or the probability of eventual capture by $x^*$ of the iterates, given that they are in its domain of attraction at some time point.

2) A simple application of the conditional Borel-Cantelli lemma allows us to infer from the above a variant of the Kushner-Clark lemma: $x(n) \to x^*$ a.s. on $\{x(n) \in B \ \text{i.o.}\}$, under the weaker condition $a(n) \to 0$.

3) As already mentioned, the above result can be shown to be strictly better than the existing results of a similar flavour.

**Important future directions are:**

1) Extension to include more general situations such as ‘Markov noise’ (see [3], Chapter 6) and asynchronous stochastic approximation (see [3], Chapter 7).

2) Our use of the Alexeev formula and more generally, our proof technique required certain regularity properties (viz., twice continuously differentiability) of $h$. It will be useful to relax this to, e.g., Lipshitz condition $h$ or even discontinuous $h$.

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**References**


