Secondary Spectrum Oligopoly Market over Large Locations

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Abstract—We investigate a secondary spectrum market where each primary owns a channel over large number of locations. Each primary sells its channel to the secondaries in exchange of a price. However, the secondaries can not transmit simultaneously at interfering locations. A primary must select a price and a set on non-interfering locations for its available channel where the availability of a channel for sale evolves randomly. The set of non-interfering locations turns out to be an independent set in the conflict graph representation of the region. The primary needs to find a strategy for each possible channel state vector. We consider node symmetric conflict graphs which arise frequently in practice when the number of locations is large (potentially, infinite). Since there is a symmetry in the interference relationship, we also consider a symmetric relationship among the joint probability distribution of the channel state vectors. We show that that a symmetric NE exists and explicitly compute it. In the symmetric NE a primary randomizes equally among the maximum independent sets at a given channel state vector. The symmetric NE exhibits several important structural differences compared to the symmetric NE strategy for small number of locations which we have obtained in our earlier works. The conflict graph representation depends on the channel state vector, thus, it is a random graph. We also empirically and theoretically investigate the expected component size in random conflict graphs which governs the computation of maximum independent sets. Our analysis shows that the mean component size is in general moderate, however, it can be high when the channel availability probability is very high. We show that with random sampling method, a primary can govern the mean component size. We numerically evaluate the ratio of the expected payoff attained by primaries in the game and the payoff attained by primaries when all the primaries collude. 

Index Terms—Game Theory, Nash Equilibrium, Secondary Spectrum Access, Quality of Service, Conflict Graph, Random Graphs, Independent Sets, Automorphism, Isomorphism, Branching Process.

I. INTRODUCTION

Secondary access of the spectrum where license holders (primaries) allow unlicensed users (secondaries) to use their channels can enhance the efficiency of the spectrum usage. However, secondary access will only proliferate when it is rendered profitable to the primaries. We investigate a spectrum oligopoly where primaries lease their spectrum to secondaries in lieu of financial remuneration. Each primary owns a channel throughout a large region consisting of several locations. The channel of a primary provides a transmission rate to a secondary depending on the state which evolves randomly and reflects the usage of the primary as well as the transmission rate due to the fading. The channel belongs to one of the states 0, 1, …, n. The secondary buys a channel which provides a transmission rate above a certain threshold. The primary can sell its channel only when the state is not 0 as at state 0 the transmission rate is below that threshold. Secondaries buy channels depending on the prices set by the primaries. This leads to a competition among the primaries.

Price competition in economics and wireless setting ignore two important properties which distinguish spectrum oligopoly from standard oligopolies: First, a primary selects a price knowing only the state of its own channel; it is unaware of states of its competitors’ channels. Thus, if a primary quotes a high price, it will earn a large profit if it sells its channel, but it may not be able to sell at all; on the other hand a low price will enhance the probability of a sale but may also fetch lower profits in the event of a sale. Second, the same spectrum band can be utilized simultaneously at geographically dispersed locations without interference; but the same band can not be utilized simultaneously at interfering locations. This special feature known as spatial reuse adds another dimension in the strategic interaction as now a primary has to cull a set of non-interfering locations, which is denoted as an independent set in the conflict graph representation of the region [1]; at which to offer its channel apart from selecting a price at every node of that set. Intuitively, a primary would like to make its channel available at an independent set of the maximum size (cardinality). However, if the competition at the largest independent set is intense, a primary may achieve higher payoff by setting high price at small independent sets (where the competition is not so intense).

We devise the problem as a game in which each primary’s strategy space consists of independent set selection strategy and the pricing strategy at each node of the independent set when the channel is available for sale. We first show that there may exist multiple asymmetric NEs. Asymmetric NEs are difficult to implement in the symmetric game that we consider (Section II-F). We, therefore, focus only on finding symmetric NEs subsequently. We prove a separation theorem (Section III-C) which entails that the NE pricing strategy at each location can be uniquely computed if the independent set selection strategy is known. By virtue of our previous work [2], [3] which characterizes pricing strategies of primaries for different transmission rates when the region has only one location (i.e. no spatial reuse). We then focus only on the independent set selection strategy.
We consider the scenario when the secondary spectrum market is operated on a large region consisting of several locations. In this setting the transmission quality of a channel may be different at different locations in the region. Thus, a primary needs to specify a strategy for each possible channel state across the network. The number of channel states and thus, the strategy space increases exponentially with number of nodes. The conflict graph representation of the region depends on the channel state across each location since a primary must select an independent set of nodes only among those nodes where the channel is available for sale. A primary is not aware of the conflict graph from which other primaries are selecting their independent sets let alone their channel states. The characterization of a symmetric NE strategy profile in the above setting is thus, more challenging. We simplify the model by assuming that the channel is either available or not (i.e. \( n = 1 \)), but the availability can differ across the nodes (Section II-C).

We focus on node symmetric or node transitive graphs (Section II-C2) [4] such as finite cyclic graph, infinite lattice graphs (e.g. infinite linear graph (infinite in both directions), infinite square graph, infinite grid graph, infinite triangular graphs) [5] which arise in practice when the region becomes large. We allow some statistical correlations which arise naturally among the channel states at different locations (Section II-C3). We show that there exists a symmetric NE strategy profile (\( \text{SP}_{sym} \)) for those graphs (Theorem 2). In the symmetric NE strategy profile, a primary randomizes uniformly among the maximum independent sets (the independent set of the highest cardinality). A primary thus only need to enumerate the maximum independent sets in order to determine \( \text{SP}_{sym} \). We also show that \( \text{SP}_{sym} \) may not be an NE in a finite linear graph which is not a node symmetric graph. We show that the symmetric NE may not be unique for a linear graph (Lemma 5) unlike in the setting where the number of locations is small which we have studied in [6].

In \( \text{SP}_{sym} \) each primary needs to enumerate the maximum independent sets. The number of independent sets grows exponentially with the nodes. However, at a given channel state vector over the region, the conflict graph may consist of several components. A primary can find maximum independent sets and \( \text{SP}_{sym} \) in each component in parallel. However, the number of maximum independent sets in a component grows exponentially with the number of nodes in the component. We, thus, investigate the size of the expected component size both analytically and empirically (Section V). Empirical result shows that the average size of components is often moderate. However, both the empirical and analytical results show that the component size can be substantially large when the channel availability probability is large. In order to control the component size we, thus, consider the setting where each primary decides to estimate the channel state at a node with a certain probability (\( p \)). A primary then sells its channel at nodes only amongst the nodes where it estimates the channel. We show that \( \text{SP}_{sym} \) is a NE strategy in this setting as well. However, if \( p \) is small, then a primary can only sell its channel at few locations which will potentially reduce the payoff. A primary thus needs to select \( p \) judiciously in order to attain a required trade-off between the computation cost and the expected payoff.

Finally, we numerically compare the expected profit obtained by the primaries using our NE strategy profile in both of the settings to the maximum possible profit allowing for collusion among primaries (Section VII). The proofs have been relegated to the archived version [7] owing to the space constraint.

**Related Literature:** Price selection in oligopolies has been extensively investigated in economics as a non co-operative Bertrand Game [8] and its modifications [9], [10]. Price competition among wireless service providers have also been explored to a great extent [11]–[14]. But all these papers did not consider the uncertainty of competition and the spatial reuse property of the spectrum oligopoly.

We now distinguish our contributions compared to our previous work [6] which is the closest to our work. In this paper we consider the setting where the channel state may be different at different locations which likely to arise when the primary owns a channel over large locations. Since a primary is not aware of the channel state vector of its competitors, thus, in our setting a primary does not know the conflict graph of other primaries from which they will select their independent sets. Thus, the collection of independent sets from which a primary selects an independent set may be different for different primaries at a given time slot since the channel state vector may be different for different primaries. Whereas in [6] the channel is either available at all locations or unavailable at any location. Thus in [6], a primary knows the conflict graph from which other primaries will select their independent sets when their channels are available. Thus, the characterization of an NE becomes significantly challenging in our setting compared to [6]. The result we obtain also significantly differs from [6]. For example, in [6] a primary can select an independent set of lower cardinalities, however, in our setting, a primary only selects the maximum independent set. Additionally, the symmetric NE is unique in a finite linear graph in [6], whereas there are infinitely many symmetric NEs in a finite linear graph in our setting.

**II. System Model**

Each primary owns a channel over a region. Unless otherwise stated, we consider that there are \( l \) number of primaries and \( m \) number of secondaries at each location throughout this paper. We, however, generalize our result for random *a priori* unknown \( m \) in Section VI. Different channels constitute disjoint frequency bands. Each primary only allows at most secondary to transmit at a given location.

**A. Transmission Rate and Channel State**

The channel of a primary provides a certain transmission rate at a location to a secondary who is granted access. Transmission rate (i.e. Shannon Capacity) at a location depends on 1) the number of subscribers of a primary that are using the channel at that location and 2) the propagation condition of the radio signal [3]. The transmission rate at a location evolves randomly over time owing to the randomness of the usage.
of subscribers of primaries and the propagation condition. We discretize the transmission rate into a number of states 0, 1, ..., n. State i provides a lower transmission rate to a secondary than state j if i < j and state\(^1\) 0 arises when the secondary can not use the channel making the channel unavailable for sale.

Let \( J \) denote the channel state vector which indicates the channel state at each node. For example, when the number of nodes are 3, then \( J = (1, 1, 0) \) is a channel state vector which indicates that the channel is in state 1, 1, and 0 at nodes 1, 2, and 3 respectively. We assume that the channels are statistically identical, specifically the probability that the channel state vector of a primary is \( J \) is \( q_J \). We also assume that the probability of the event where the channel state is 0 at every location is non-zero i.e.

\[
q_J > 0 \quad \text{when} \quad J = \{0, 0, \ldots, 0\} \tag{1}
\]

Let the set of channel state vectors be \( \mathcal{P} \). Thus, the cardinality of \( \mathcal{P} \) is \( (n + 1)^{|V|} \) where \( |V| \) is the number of locations.

### B. Conflict Graph

Each primary owns a channel over a broad region consisting of several locations. Typically, secondary users can not transmit simultaneously using the same channel at adjacent locations due to interference. In order to sell its channel a primary needs to find a set of locations which do not interfere with each other. Wireless networks have been traditionally modeled as conflict graphs (Figures 1, 2) in most of the existing literature including in several seminal papers [15], [16]. Let \( G = (V, E) \) be the overall conflict graph of the region where \( V \) is the set of nodes and \( E \) is the set of edges; an edge exists between two nodes if transmission at the corresponding locations interfere. In a conflict graph, the set of nodes in which no edge exists between any pair of nodes is called an independent set (Fig. 1). Thus, secondaries at all nodes in an independent set, can transmit simultaneously using the same channel without any interference.

Note that when the channel of a primary is at state 0 at a node, then the primary can not sell its channel at that node. Thus, a primary ought to offer its channel at a set of non-interfering locations among the locations where the channel is available for sale (i.e. the state of the channel is not 0). Let \( G_J = (V_J, E_J) \) be the conflict graph representation of the channel state vector \( J \): \( V_J \) is the set of nodes (locations) where the channel is available for sale at channel state vector \( J \) of a primary and \( E_J \) is the set of edges in \( G \) between the nodes of \( V_J \). \( G_J \) is obtained by removing the nodes and the edges corresponding to those nodes from \( G \) where the channel is not available i.e. the channel is at state 0. Thus, \( G_J \) is a subgraph of \( G \). Figure 3 represents a conflict graph \( G \) of a region and the conflict graph \( G_J \) when the channel state vector is \( J \). A primary needs to select an independent set from \( G_J \) when the channel state vector is \( J \).

\(^1\)Generally a minimum transmission rate is required to send data. State 0 indicates that the transmission rate is below that threshold due to either the excessive usage of subscribers of primaries or the transmission condition.

![Fig. 1: Figure in (a) shows a wireless network with \( M \) number of locations. There are \( m = 2 \) secondaries at each location. Signals at locations 1 and 2 and 2 and 3 interfere with each other, but signals at locations 1 and 3 do not interfere.](image)

![Fig. 2: The rectangle represents a shop in a shopping complex or a department in a university campus. Circles 1, 2, 3, 4 are the ranges of Wireless access points. Each circle corresponds to a node in the conflict graph. Since ranges of Wireless access points intersect with each other, thus there exists an edge between every pair of nodes.](image)

### C. Simplifying Assumptions

1) \( n = 1 \): The conflict graph representation of the region depends on the channel state vectors. Since the conflict graph representation can be different for different channel state vectors \( G_J \) may not be equal to \( G_K \) when \( J \neq K \), thus, a primary does not know the conflict graphs from which its competitors are selecting their independent sets. Thus, the collection of independent sets from which a primary selects its independent set may be different for different primaries. Additionally, the strategy space \( \mathcal{P} \) \( \{(|\mathcal{P}| = (n + 1)^{|V|}\) increase exponentially with the number of nodes. Thus, obtaining an NE in this setting is challenging. In order to simplify the setting, we consider
Fig. 3: The conflict graph for the overall region is \( G \) which corresponds to the situation where the channel is available at all nodes in the region. \( G_j \) is the conflict graph when the channel state vector is \( J = (j_1, j_2, j_3, 0, 0, 0) \) where \( j_i \geq 1, i = 1, 2, 3 \). Since the channel states are 0 at nodes 4, 5, and 6, thus, \( G_j \) is obtained by removing those nodes and the edges corresponding to those nodes.

**Assumption 1.** \( n = 1 \) i.e. the channel is either available (i.e. at state 1) or not available (i.e. at state 0) at each node, but still the channel state can be different at different nodes.

Note that even though \( n = 1 \), the cardinality of strategy space \( \mathcal{P} \) is \( 2^V \) which is still exponential in the number of nodes and the conflict graph representation will be different for different channel state vectors.

2) **Node Symmetric Graphs:** We consider large size wireless networks. As an analytical abstraction, we mainly consider infinite size conflict graphs. In large conflict graphs, there is an inherent symmetry in the interference relations among the nodes in the network. We, therefore consider node symmetric graphs, which in the literature is also known as node transitive graphs [4].

First, we provide a formal definition of node symmetric graph. Towards that end, we first define an automorphism in a conflict graph \( G \). We denote \( V(G) \) as the set of nodes of \( G \).

**Definition 1.** An automorphism is a bijective mapping \( F : V(G) \rightarrow V(G) \) such that nodes \( F(a) \) and \( F(b) \) are adjacent\(^2\) if and only if nodes \( a, b \) are adjacent in \( G \).

In an automorphism, the nodes can be renumbered such that it maintains the adjacency between the nodes. For example, consider a linear graph consisting of 4 nodes. Fig. 4 shows an automorphism on this graph. Now we are ready to define the node symmetric graph.

**Definition 2.** [4] In a node symmetric graph, for every pair of vertices \( a \) and \( b \) of \( G \), there is some automorphism \( F : V(G) \rightarrow V(G) \) such that \( F(a) = b \).

For a graph to be node symmetric every node should be mapped to every other node through an automorphism. Informally, in a node symmetric graph the graphs looks the same from each node.

For example cyclic graph is a node symmetric graph. But linear graph with 4 nodes is not a node symmetric graph since

\(^2\)In an undirected graph, two nodes are adjacent iff there is an edge between them.

Fig. 4: Left hand figure shows a linear graph with 4 nodes. Right hand figure an automorphism where \( F(1) = 4, F(2) = 3, F(3) = 2, F(4) = 1 \). However, there is no automorphism between nodes 2 and 1. If there is an automorphism such that \( F(2) = 1 \), then by the property of automorphism nodes \( F(2) \) and \( F(3) \) should be adjacent and nodes \( F(2) \) and \( F(1) \) also should be adjacent, but since \( F(2) = 1 \), thus, either \( F(3) \) or \( F(1) \) will not be adjacent to node \( F(2) \) since node 1 only has one degree in \( G \).

Fig. 5: Infinite linear graph with no end-points: each node has degree 2. There is no automorphism between nodes 1 and 2 (Fig. 4).

Now, we provide some examples of infinite node symmetric graphs which resemble the conflict graphs of large wireless networks.

- Infinite linear graph with no end points (Fig. 5): This is an abstraction of the conflict graph of the network of a large number of wireless access points arranged in a linear fashion.
- Infinite square graphs (Fig. 6): This is an abstraction of the conflict graph of wireless networks in a large region with square cells.
- Infinite grid graphs (Fig. 7): This is an abstraction of the conflict graph of a large shopping mall.
- Infinite triangular graphs (Fig. 8): This is an abstraction of the conflict graph representation of large number of hexagonal cells [17].

There are also several commonly observed node symmetric conflict graphs which are finite. For example, cyclic graph of any size is a node symmetric graph. Cyclic conflict graph represents a collection of wireless access points arranged in a circular fashion, possibly circumambulating a city or ring size road. The complete graphs \(^3\) are also node symmetric graphs. We find a symmetric NE in a node symmetric graph irrespective of whether it is finite or infinite (Theorem 2).

Note that the commonly observed conflict graphs of small networks are mean valid graphs which we analyze in our previous paper [6]. **These graphs may not be node symmetric graphs.**

3) **Probability Distribution of Channel State Vectors:** In the previous setting, we consider an extreme case where the channel state is identical across each location. In a large network, the channel states will be different. However, the channel states are often spatially proximal. Since the graph is large, like the interference relationship we expect that the statistical correlation pattern would also exhibit some symmetry. We consider one such symmetric relationship among the channel states across the network which arise naturally.

First, we define an isomorphism between two graphs:

\(^3\)In a complete graph a node has edge with every other node.
Definition 3. Two graphs $G$ and $H$ are isomorphic to each other if there is a bijective mapping $F : V(G) \to V(H)$ such that any two vertices $F(a), F(b)$ are adjacent in $H$ if and only if $a, b$ are adjacent in $G$.

Informally, if two graphs look alike subject to renumbering of nodes, then they are isomorphic to each other. Note that automorphism is a special case of isomorphism which occurs when $H = G$ (Definition 1).

We assume that

Assumption 2. $q_J$ and $q_K$ are identical whenever the $G_J$ and $G_K$ are isomorphic to each other.

Intuitively, since $G_J$ and $G_K$ are alike subject to the renumbering of nodes, we therefore expect $q_J = q_K$. We show in [7] that the above assumption implies that

Lemma 1. The probability that a channel of a primary is in state $1$ at a given location is the same across the network.

However, the converse of the above result not true in general.

We now provide some examples of joint probability distributions which arise in practice and satisfy Assumption 2.

Independent and identically distributed channel states: The state of the channel is $i = 1$ w.p. $q$ at a given location independent of the channel states at other locations. At a given channel state vector $J$, if the channel is available at $n_j$ number of nodes, then $q_J = q^n_i(1 - q)^{|V| - n_j}$. When $G_J$ and $G_K$ are isomorphic, then both contain the same number of nodes, thus, the number of locations where the channel state is $1$ (0, resp.) are the same in channel state vectors $J$ and $K$. Hence, the probability distributions $q_J$ and $q_K$ are identical whenever $G_J$ and $G_K$ are isomorphic.

Correlated Channel states: We now show that Assumption 2 can accommodate statistical correlations across the channel states at different nodes. We provide an example in a small node symmetric graph. Consider a linear graph with 2 nodes such that $q_{(1,0)} = q_{(0,1)}$. Since $G_{(0,1)}$ and $G_{(1,0)}$ are the only possible isomorphic graphs in this case, thus, the above joint probability distribution satisfies Assumption 2. Now, if $q_{(1,1)} > q_{(1,0)} = q_{(0,1)}$ and $q_{(0,0)} > q_{(1,0)} = q_{(0,1)}$, then, the channel states are not independent$^4$. Thus, Assumption 2 allows correlation among the channel states across the locations. Also note that the above probability distributions commonly arise in practice. This is because when the channel is in state 1 (0 respv.) at one location, then there is a higher probability that the channel is in state 1 (0 respv.) compared to state 0 (1 respv.) at another location.

The joint probability distributions of random variables associated with spatial locations and exhibiting correlations are often represented as Markov Random Field. We provide a formal definition of Markov random field in Appendix A and show that the Markov random field modeling of channel states where the channel states in neighboring locations are correlated, satisfy Assumption 2 under some additional assumptions which naturally arise (Lemma 8 in Appendix A).

D. Decision of the Secondaries

Secondaries buy channel from the primaries that offer the lowest price. Each secondary achieves an utility by using an channel. We assume that the secondaries are statistically identical, in particular, they attain the same utility $v$ from a channel which is available for sale. Thus, a secondary does not buy a channel which is priced above $v$.

E. Strategy and Payoff of Each Primary

For each channel state vector $J \in \mathcal{P}$ a primary selects: a) an independent set of the conflict graph $G_J$ where it will sell its channel; b) a price at every node of that independent set.

A primary arrives at its decision with the knowledge of its own channel state vector $q_J$ but without knowing the channel state vector of other primaries. A primary however knows $l, m, v, G$ and $q_{l}, J \in \mathcal{P}$. Secondaries strictly prefer a channel which induces lower price compared to the higher price as discussed in Section II-D. The ties among channels with identical prices are broken randomly and symmetrically among the primaries. We formulate the decision problem of primaries as a non-cooperative game with primaries as players.

Definition 4. A strategy of a primary $i$ $\psi_{i,J}$ provides the probability mass function (p.m.f) for selection among the independent sets (I.S.s) and the price distribution it uses at each node, when its channel state vector is $J$. $S_i = (\psi_{i1}, ..., \psi_{ij}, p_i)$ denotes the strategy of primary $i$, and $(S_1, ..., S_l)$ denotes the strategy profile of all primaries. $S_{-i}$ denotes the strategy profile of primaries other than $i$.

Each primary incurs a transition cost $c$ at each location where it is able to sell its channel. If primary $i$ selects a price $x$ at node $s$, then its payoff at node $s$ is$^5$

$^4$Suppose that the channel is in state 1 at node $i$ w.p. $q_1$, independent of the channel state at other location, then, $q_1 = q_{01}$ implies that $q_{11} = q_2$. Now, $q_1(1 - q_1)$ can not be less than both $q_1^2(= q_{11})$ & $(1 - q_1)^2(= q_{00})$, hence, independent channel states can not satisfy the above joint distribution.

$^5$Note that if $Y_s$ is the number of channels offered for sale at a node $s$, for which the prices are upper bounded by $v$, then those with min$(Y_s, m)$ lowest penalties are sold since secondaries select channels in the increasing order of penalties.
players and profile, then the strategy profile is an asymmetric NE.

The payoff of a primary over an independent set is the sum of payoff that it gets at each node of that independent set. Thus, if a primary is unable to sell at any node of an independent set, then its payoff is 0 over that independent set.

**Definition 5.** \( u_{i,J}(\psi_{i,J},S_{-i}) \) is the expected payoff when primary \( i \)'s channel state vector is \( J \) and selects strategy \( \psi_{i,J}(\cdot) \) and other primaries use strategy \( S_{-i} \).

**F. Solution Concept**

We seek to obtain a Nash Equilibrium (NE) strategy profile which we define below using \( u_{i,J} \) (Definition 5), \( \psi_{i,J} \) and \( S_{-i} \) (Definition 4):

**Definition 6.** [8] A Nash equilibrium \((S_1, \ldots, S_l)\) is a strategy profile such that no primary can improve its expected profit by unilaterally deviating from its strategy. So, with \( S_i = (\psi_{i1}, \ldots, \psi_{i|P|}) \), \((S_1, \ldots, S_l)\), is a Nash equilibrium (NE) if for each primary \( i \) and channel state vector \( J \)

\[
u_{i,J}(\psi_{i,J},S_{-i}) \geq u_{i,J}(\tilde{\psi}_{i,J},S_{-i}) \, \forall \, \tilde{\psi}_{i,J}.	ag{2}\]

An NE \((S_1, \ldots, S_l)\) is a symmetric NE if \( S_i = S_j \) for all \( i, j \).

If \( S_i \neq S_k \) for some \( i, k \in \{1, \ldots, l\} \) in an NE strategy profile, then the strategy profile is an asymmetric NE.

In a symmetric game, as the one we consider, it is difficult to implement an asymmetric NE. For example, if there are two players and \((S_1, S_2)\) is an asymmetric NE i.e. \( S_1 \neq S_2 \), then \((S_2, S_1)\) is also an NE due to the symmetry of the game. The realization of such an NE is only possible when one player knows whether the other is using \( S_1 \) or \( S_2 \). But, apriori coordination among players is infeasible as the game is non-co-operative.

Note that if \( m \geq l \), then primaries select the highest price \( v \) at each node and will select one of the maximum independent sets of \( G_J \) at channel state vector \( J \) with probability 1. This is because, when \( m \geq l \), then the channel of a primary will always be sold at a location. Hence, a primary will be always able to sell its channel at the highest possible price. Henceforth, we will consider that \( m < l \).

**III. Initial Results, Multiple NESes, and A Separation Result**

**A. Results Of One-shot Single Location Game**

Now, we briefly summarize the main results of the game when it is limited to only one location, which we have studied in [2], [3]. Since there is only node, thus, the channel state vector reduces to a scalar and we denote \( q_1 \) when the channel is in state \( j = 0, 1 \) at that node. Note that there is no spatial reuse constraint in this setting, thus a primary’s decision is only to select a price.

We start with following definitions. Let \( w(x) \) be the probability of at least \( m \) successes out of \( l-1 \) independent Bernoulli trials, each of which occurs with probability \( x \). Thus,

\[
w(x) = \sum_{i=m}^{l-1} \binom{l-1}{i} x^i (1-x)^{l-1-i}.	ag{3}\]

Note that \( w(\cdot) \) is continuous and strictly increasing in \([0, 1]\), so its inverse exists.

Now, let for \( 1 \leq j \leq n \),

\[
\tilde{p} - c = (v - c)(1 - w(q_1))
\]

**Lemma 2.** [2], [3] A NE strategy profile must be symmetric and each primary’s price selection strategy is the following

\[
\phi(x) = 0, \text{if } x < \tilde{p}
\]

\[
\frac{1}{q_1} (w^{-1}(x - \tilde{p}) - c), \text{if } v \geq x \geq \tilde{p}
\]

\[
1, \text{if } x > v.
\]

**Theorem 1.** [2], [3] The strategy profile, in which each primary randomizes over the prices in the range \([\tilde{p}, v]\) using the continuous distribution function \( \phi(\cdot) \) (Lemma 2), is the unique NE strategy profile. The expected payoff that a primary attains at every value within the interval \([\tilde{p}, v]\) is \( \tilde{p} - c \).

**B. Multiple Asymmetric NESes**

We first show that there can be multiple NESes in this game unlike in the single location game studied in [2] [3]. Consider the linear conflict graph depicted in Fig. 1 with 2 nodes, 2 primaries and 1 secondary. Note that a linear graph with 2 nodes is a node symmetric graph. Thus, the following result will also show that there exists multiple asymmetric NESes in node symmetric graphs.

We need to specify strategy at each possible channel state vector. We also consider that the channel state of a primary is 1 at a given location w.p. \( q_1 \) independent of the channel state at other location. The following strategy profiles are NE strategy profiles: i) When the channel state vector is \((0, 1)\) \((1, 0)\) respv. then a primary selects node 2 (1 respv.) w.p. 1 and selects the single location pricing strategy stated in Theorem 1 with \( q_1, q_0 \) in place of \( q_1^0 \). When the channel state vector is \((1, 1)\) then primary 1 (primary 2 respv.) selects node 1 (node 2 respv.) w.p. 1 and selects price \( v \) w.p. 1.

ii) When the channel state vector is either \((0, 1)\) or \((1, 0)\) then the strategy profile is the same as before. When channel state vector is \((1, 1)\) then primary 1 (primary 2 respv.) selects node 2 (node 1 respv.) w.p. 1 and selects price \( v \) w.p. 1.

Note that NE strategy profiles cited above are asymmetric. The game is a symmetric one since primaries have the same action sets, payoff functions and their channels are statistically identical. In a symmetric game, we have already discussed in Section II-E that implementing an asymmetric NE is difficult. We therefore focus on finding a symmetric NE and investigate whether it is unique. Clearly, for any symmetric NE, we can represent the strategy of any primary as \( S = (\psi_{1}(\cdot), \psi_{2}(\cdot), \ldots, \psi_{|P|}(\cdot)) \) where we drop the index corresponding to the primary.

\( q_1^0 q_0 \) is the probability that the channel state vector is either \((0, 1)\) or \((1, 0)\).
C. A Separation Result

We now observe that the NE price selection at a node in an independent set can be uniquely computed using the single location NE price selection strategy stated in Section III-A once the independent set selection strategy is known.

Lemma 3. Suppose, under a symmetric NE, each primary offers its available channel at node $a$ for sale w.p. $\alpha_a$. Then, the unique NE price distribution of each primary is the d.f. $\phi(\cdot)$ as described in Lemma 2 with $\alpha_a$ in place of $q_1$ at node $a$.

We next obtain the expression for $\alpha_a$. We first introduce some notations: Let $I_J$ be the set of independent sets of the graph $G_J$. Let $P_a$ be the set of channel state vectors where the channel state is $j$ at node $a$.

Definition 7. Let $\beta_J(I)$ be the probability with which the independent set $I \in I_J$ is selected by a primary, under a symmetric NE strategy when the channel state vector is $J$.

Note that though $\beta_J(I)$ depends on the symmetric NE strategy, we do not make it explicit in the notation in order to keep the notational simplicity. Thus,

$$\alpha_a = \sum_{I \in I_J} \sum_{a \in P_a} q_J \beta_J(I) \tag{6}$$

Note from Theorem 1, the expected payoff at a node $a$ to a primary is $\tilde{p}_a - c$ where

$$\tilde{p}_a - c = (v - c)(1 - w(\alpha_a)) \tag{7}$$

If $\alpha_a = 0$ i.e. the available is channel is offered for sale at node w.p. 0 at node $a$, then the payoff is 0.

Since the price selection strategy of a primary is unique given the independent set selection strategy $\{\beta_J(I)\}$ (by Lemma 3), henceforth, we only focus on independent set selection probabilities which provide the node selection probabilities as defined in (6).

IV. SYMMETRIC NE STRATEGY PROFILE

We, first, obtain a symmetric NE strategy profile (Theorem 2). We then show that the NE strategy has an important structural difference compared to the NE strategy in the setting where the channel state is the same throughout the same which we have studied in [6] (Lemmas 4, 5).

We first start with introducing a notation. Let $I_{\text{max},J}$ be the set of maximum independent sets (i.e. independent sets of highest cardinalities) of the graph $G_J$.

Strategy Profile ($SP_{\text{sym}}$): A primary selects each of the independent set within the set $I_{\text{max},J}$ with probability $\frac{1}{I_{\text{max},J}}$ and select other independent sets with probability 0 at channel state vector $J$.

We show in [7] that

Theorem 2. The Strategy profile $SP_{\text{sym}}$ is an NE strategy profile.

A primary only needs to find the maximum independent sets in order to find the NE strategy profile $SP_{\text{sym}}$. In contrast to $SP_{\text{sym}}$, a primary may select an independent set which is not a maximum independent set in the scenario where the channel state is identical across the locations which we have studied in [6]. Note that in $SP_{\text{sym}}$ a primary puts equal weight on each of the maximum independent sets in $G_J$. Hence, a primary needs not communicate with other primaries to obtain its strategy. Hence, $SP_{\text{sym}}$ is easy to implement.

Lemma 4. Expected payoff at every node is the same under $SP_{\text{sym}}$.

Intuitively, since the graph is node symmetric, each node belongs to the same number of maximum independent sets, thus a channel is offered with the same probability at every node under $SP_{\text{sym}}$; thus, the expected payoff is the same at every node. Since each node is selected with the same probability, hence there is an equity in secondary access of the channel amongst different nodes as opposed to that scenario when the channel state remains the same at each location [6]. Note that the setting where the channel state remains the same arises when the number of locations are small which we have studied in [6]. Thus, Lemma 4 entails that though when the secondary market operates over a small number of location, there is an inequity in the secondary access of the channels; as the secondary market expands the inequity is removed.

We show in [7] that unlike in the scenario where the channel state is the same across the network [6], the symmetric NE may not be unique in linear conflict graph in this setting.

Lemma 5. There may exist infinitely many symmetric NEs in the linear conflict graph.

Our analysis also reveals that the symmetry in interference relations among the nodes is required for $SP_{\text{sym}}$ to be an NE.

Lemma 6. $SP_{\text{sym}}$ may not be an NE for a finite linear graph which is not a node symmetric graph.

V. COMPUTATIONAL COMPLEXITY

In $SP_{\text{sym}}$, a primary needs to enumerate the maximum independent sets at a given channel state vector. In general, the number of maximum independent sets scales exponentially with the number of nodes. But if a graph consists of disjoint components, then a primary can compute the maximum independent sets of each component and compute the strategy profile in each component in parallel. Hence, the size of the component will govern the computation time.

The conflict graph of a primary depends on the channel state vector which evolves randomly. Hence, the conflict graphs are random graphs. Thus, it is important to find the average size of a component in a conflict graph which will govern the average computation time of maximum independent sets. In the following, we provide a bound on the expected size of a component for some node symmetric graphs that arise in practice. We also discuss how primaries can govern the component size using random sampling technique (selecting each node w.p. $p$). Throughout this section, we consider that the channel states are I.I.D. where the channel state is 1 at a given location w.p. $q$.

Let $\Delta$ be the degree of a node. We consider those node symmetric graphs where $\Delta$ is finite. Nodes in most of the
conflict graphs that we have discussed in Section II-C2 have finite degrees. For example, in cyclic graph (of any size) \( \Delta = 2 \), in infinite linear graph \( \Delta = 2 \), in infinite square graph (Fig. 5), \( \Delta = 4 \) (Fig. 6), in infinite grid graph \( \Delta = 8 \) (Fig. 7), in infinite triangular graph \( \Delta = 6 \) (Fig. 8).

We find out the expected size of a component \( C \) originating from node \( a \) in a conflict graph \( G_J \). Since the graph is a node symmetric graph, hence the expected size of a component originating from any other node will be the same. Each node has an expected degree of \( q\Delta \). The component \( C \) grows when \( G_J \) contains neighbors of node \( a \), the neighbors of the neighbors of node \( a \), and so on. Thus, the growth of \( C \) can be compared to the Galton-Watson branching process [18] where each individual gives birth to \( q\Delta \) number of children on average. The only difference in our approach to the Galton-Watson process is that the number of nodes added each step may be smaller as some of the neighbors of a node may already be in \( C \), thus, reducing the number of neighbors that can be added in \( C \). Thus, the expected size of \( C \) can be upper bounded by the expected number of total descendants in Galton-Watson process [18]. Hence, the upper bound of expected size of \( C \) is obtained from [18]

**Lemma 7.** \( E(C) \leq \frac{1}{1 - q\Delta} \) if \( q\Delta < 1 \).

A primary can not control \( q \), hence, the component size (and thus, the computational complexity) can be large for higher \( q \). Thus, a primary may estimate its channel quality only at a subset of the locations of the region instead of the whole region and sell its channel only among the locations where it knows the transmission quality in order to minimize the computation cost. Equivalently, a primary will consider that the channel state is 0 at locations where it does not estimate its channel quality. In one simplistic setting which we consider, each primary computes the transmission quality at a node w.p. \( p \) independent of the other nodes. A primary does not know the nodes where its competitors are estimating their channel states. But, it knows \( p \). Thus, a primary is aware of the fact that the channel state is 1 at any given node of its competitor w.p. \( pq \) independent of the channel states at other locations. Thus, the probability distribution satisfies Assumption 2. Hence, the strategy profile \( SP_{sym} \) will be a symmetric NE strategy profile in this setting where the channel states are i.i.d. and the channel is in state 1 at a given location w.p. \( pq \) instead of \( q \). Thus, from Lemma 7 the expected size of component is now upper bounded by

\[
E(C) \leq \frac{1}{1 - pq\Delta} \quad \text{if} \quad pq\Delta < 1
\]  

(8)

Note that the above procedure also decreases the measurement and estimation cost, since a primary only estimates the channel states at a randomly selected subset of locations.

Note that the right hand side in (8) increases as \( pq\Delta \) increases. If a primary selects lower (higher, respv.) \( p \) the expected component size will decrease (increase, respv.), and thus, the computation complexity will decrease (increase, respv.). The bound in (8) also decreases (increases, respv.). However, the expected payoff of a primary will also decrease (increase, respv.) with decrease in \( p \) (increase, respv.) since the number of nodes where a primary can potentially sell its channel also decreases (increases, respv.). Hence, a primary has to judiciously select \( p \) in order to achieve a desired trade-off between the computation complexity, and the expected payoff.

We now empirically investigate the variation of the mean size of the largest component with the number of nodes and the parameter \( pq \). For each value of \( pq \), we generate a certain number of random graphs. We compute the average of the largest component over 25 samples. The number 25 has been chosen since the average converges in 25 samples. Figure 9 shows the variation of the mean size of the largest component as the number of nodes increases. Figure 9 reveals that the growth of the average size of the largest component in a square graph is linear (not exponential) with the number of nodes whereas the growth of the mean size of the largest component in a linear graph is very slow with the number of nodes. Additionally, when \( pq = 0.5 \), the upper bound in (8) is infinite both for square and linear graph, however, Fig. 9 shows that the expected size of the largest component is moderate in the square graph as well as in the linear graph even when the number of nodes are large. Fig. 10 reveals that when \( pq \) exceeds a threshold the mean size of a largest component increases substantially in both linear conflict graph and square conflict graph. However, Fig. 10 reveals that the upper bound computed in (8) is loose. For example, when \( 0.25 \leq pq \leq 0.6 \), the upper bound in (8) is infinite for square graph, however, Fig. 10 shows that the average size of the largest component is moderate. In the linear graph, the mean size of the largest component is small even when \( 0.5 \leq pq \leq 0.75 \) whereas the upper bound computed in (8) is infinite when \( 0.5 \leq pq \leq 0.75 \).

**VI. Random Demand**

Till now we have assumed that the number of secondaries \( m \) is constant at each node. But our analysis will readily
Definition 8. The efficiency of NE is the ratio of the total expected welfare of primaries and the optimal value of social welfare ($R_{OPT}$) which is obtained when all the primaries collude.

$$w(x) = \min \{ \max(Z), l-1 \} \kappa \sum_{i=1}^{l-1} \binom{l-1}{i} x^i (1-x)^{l-i-1} \quad (9)$$

VII. NUMERICAL EVALUATIONS

We numerically study the impact of competition on the payoffs of the primaries in the scenarios which we consider. Towards that end, we compare the payoff under the symmetric NE strategy, $R_{M,NE}$, with the maximum possible value of social welfare, $R_{OPT}$ which is obtained when all the primaries collude.

$$R_{M,NE} = \text{Number of Primaries} \times \text{Expected payoff of each primary}$$

Definition 8. The efficiency of NE is the ratio of the total expected payoff of primaries and the optimal value of social welfare ($R_{OPT}$).

In other words, efficiency ($\eta_M$) = $\frac{1}{R_{M,OPT}}$. $R_{M,NE}$.

Fig. 11 reveals that $\eta_M$ increases with increase in $m$. This is because when $m$ is low, competition becomes intense and primaries select lower prices. But if they collude with each other, they still can offer highest price and only select the independent sets of the largest cardinalities in both of the settings, which lead to high payoff.

VIII. FUTURE WORK

The characterization of an NE in the setting where the available channel may belong to multiple states (i.e. $n > 1$) remains open. The analytical results and tools that we have provided in this paper may provide the basis for developing a framework for this problem.

REFERENCES


APPENDIX

A. Markov Random Field

1) Background: A Markov random field is a graphical model which represents the joint probability distributions of random variables having Markov property. It is represented by an undirected graph $H = (V, E)$ in which the nodes $V$ represents the random variables. The edges $E$ encodes the dependencies among the random variables in the following way: if $N(A)$ is the set of neighbors of $A$, then in a Markov random field [19],

$$A \perp \text{other random variables} \mid N(A)$$

Figure 12 provides a cyclic Markov random field. Here, $A \perp C \mid B, D$. The channel states in a conflict graph are random variables whose values are either 0 or 1. Since the channel states at adjacent locations are likely to be correlated, we model the correlation amongst the adjacent locations in...
the conflict graph using the Markov Random field where the nodes in the Markov random field represent the channel states of the corresponding nodes of conflict graph $G$. Figure 12 represents a Markov random field when the conflict graph is a cyclic graph with 4 nodes and the values of the random variables $A, B, C, D \in \{0, 1\}$ represent the channel states at nodes $A, B, C, D$ of conflict graph $G$ respectively.

We now discuss the joint probability distribution in the Markov random field. Markov random fields provide a compact representation of the joint probability distribution in terms of product of potential functions. Potential functions are defined on the set of maximal cliques, $C$, in the graphical representation of the Markov random field $H$. A potential function $\zeta_C(\cdot)$ represents the values of the random variable of the maximal clique $C \in C$. For example, in figure 12 the set $\{AB\}$ is a maximal clique, thus, $\zeta_{AB}(a,b)$ denote the value of the potential function when the random variables $A = a$ and $B = b$, $a, b \in \{0, 1\}$. Note that $\zeta_C(\cdot)$ is defined on the vector $c$ which represents the values of the random variables represented by nodes in the clique $C$. Formally, the probability of the channel state $J$ is given by:

$$q_J = \frac{1}{Z} \prod_{C \in C} \zeta_C(c_J) \quad (10)$$

where $Z$ is a normalization factor and $c_J$ denote the channel states in clique $C$ when the overall channel state vector is $J$.

For example, in figure 12 the set of maximal cliques $C$ is $AB, BC, CD, DA$. The joint probability distribution is given by

$$P_{A,B,C,D}(a,b,c,d) = \frac{1}{Z} \zeta_{AB}(a,b)\zeta_{BC}(b,c)\zeta_{CD}(c,d)\zeta_{DA}(d,a)$$

Since $A, B, C, D$ only take values in $\{0, 1\}$, we can represent $\zeta$ as a matrix where $\zeta_{AB}(a,b)$ denote the value of the $(a,b)$th position of the matrix. For example, $\zeta$ can be the following:

$$\zeta_{AB} = \zeta_{BC} = \zeta_{CD} = \zeta_{DA} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 1 \end{bmatrix} \quad (11)$$

In Figure 13, the maximal cliques are $(ABC, CD)$. Hence, the joint probability distributions are

$$P_{A,B,C,D}(a,b,c,d) = \frac{1}{Z} \zeta_{AB}(a,b)\zeta_{CD}(c,d) \quad (12)$$

**Definition 9.** The Markov random field representation of random variables is symmetric if i) the maximal cliques are of identical sizes and ii) suppose $c_1$ corresponds to the channel state vector of maximal clique $C_1$ and $c_2$ corresponds to the channel state vector of maximal clique $C_2$, then

$$\zeta_{C_1}(c_1) = \zeta_{C_2}(c_2) \quad (13)$$

for every $c_1$ and $c_2$ such that $c_1$ and $c_2$ contain the same number of 1s (and thus, the same number of 0s since $C_1, C_2$ are of same sizes).

(11) provides an example of potential functions which are symmetric and identical. But potential functions in Fig. 13 can not be symmetric since the sizes of the maximal cliques are different.

Now, we are ready to provide an example which satisfies Assumption 2.

2) Result:

**Lemma 8.** The probability distributions on the channel states satisfy Assumption 2 if

i) The channel states constitute a Markov random field,

ii) The graphical representation of the Markov random field $H$ is the same as the node symmetric graph $G$,

iii) The Markov random field relation is symmetric, and

iv) There are fixed integers $r_1, r_2, \ldots$ such that every clique containing $j \geq 1$ number of nodes is a subset of identical $(r_j)$ number of maximal cliques in $G$.

First, it is easy to discern that the condition (iv) is satisfied by a large class of node symmetric conflict graphs including cyclic graph, infinite linear graph (Fig. 5), infinite square graph (Fig. 6), infinite grid graph (Fig. 7), infinite triangular graph (Fig. 8). For example, in the infinite triangular graph (Fig. 14), a clique containing 3 nodes is a maximal clique and hence, it is a part of only 1 maximal clique; any clique containing 2 nodes is a subset of 2 maximal cliques; a single node is a part of 6 maximal cliques. We prove the above lemma in [7].

\footnote{In a node symmetric graph, the maximal cliques are of the same size}