Localization in Internet of Things Network: Matrix Completion Approach

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Abstract—In this paper, we propose a matrix completion algorithm for Internet of Things (IoT) localization. In the proposed algorithm, we recast Euclidean distance matrix completion problem as an unconstrained optimization in smooth Riemannian manifold and then propose a nonlinear conjugate gradient method on this manifold to reconstruct Euclidean distance matrix. The empirical results show that the proposed algorithm is effective and also outperforms state-of-the-art matrix completion algorithms both in noise and noiseless scenarios.

I. INTRODUCTION

Recently, Internet of Things (IoT) has attracted much attention due to its wide variety of applications, such as healthcare, surveillance, and environmental monitoring. One of essential components in the IoT is wireless sensor network, in which environmental data (e.g., temperature, humidity, and object movements) is collected and processed using hundreds of sensor nodes [1], [2]. In order to respond and react to the environmental data, location information of sensor nodes should be available at the base station (a.k.a., data center, sensor fusion, access point) [3]. Since the action in IoT networks, such as fire alarm, energy transfer, emergency request, is made primarily on the data center, an approach to identify the location information of whole nodes at the data center is of importance. In this approach, dubbed as localization at data center, each sensor node measures the distance information of adjacent nodes and then send it to the data center. Using the obtained distance information, data center constructs a map of sensor nodes [2].

In order to perform the localization at the data center, pairwise distance information between each sensor pair should be provided. It has been shown that if absolute locations of a few sensor nodes (often called anchor nodes) are provided, one can find out the location information of sensor nodes accurately [3]. Main problem of the localization at the data center is that the data center might not have enough distance information to identify the locations of sensor nodes. In general, one cannot recover the original Euclidean distance matrix $D$ from a subset of its entries since there are infinitely many completion options for the unknown entries. However, it is now well-known that if $D$ is a rank-$k$ matrix, then $D$ can be recovered from $D_{obs}$ with reasonably number of measured entries which is not far away from information theoretic limit. [4].

The problem to recover a low-rank matrix $D$ from the small number of known entries is described as

$$
\min_{\tilde{D} \in \mathbb{R}^{n \times n}} \text{rank}(\tilde{D}),
\text{s.t.} \quad \mathcal{P}_E(\tilde{D}) = D_{obs}.
$$

where $\mathcal{P}_E$ is the sampling operator defined by
$$
|\mathcal{P}_E(A)|_{ij} = \begin{cases} A_{ij} & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}
$$

In our case, $E = \{(i,j) : \|x_i - x_j\|_2 \leq r\}$ is the set of the observed indices where $r$ is the radio communication range. Due to the non-convexity of the rank function, it is computationally infeasible to solve this problem. Recently, many efforts have been made to transform the problem into more tractable form. One popular way is to use the nuclear norm $\|D\|_*$ instead of $\text{rank}(D)$. The nuclear norm is the sum of singular values (i.e., $\|X\|_* = \sum_i \sigma_i$). Note that the nuclear norm minimization is the tightest convex relaxation of the (unrealizable) rank minimization problem in (1).

Although this approach can be solved by the semidefinite programming (SDP) [4], computational overhead is still burdensome for the practical systems.

Another approach is to use the least-squared minimization:

$$
\min_{\tilde{D} \in \mathbb{R}^{n \times n}} \frac{1}{2} \|\mathcal{P}_E(\tilde{D}) - D_{obs}\|_F^2,
\text{s.t.} \quad \text{rank}(\tilde{D}) \leq \eta,
$$

where $\eta$ is the rank of the original matrix. Since the rank constraint in (1) is replaced by the Frobenius norm based cost function, this approach is effective in the noisy scenario. In recent years, various approaches to find a solution of (2) have been suggested. In [5], alternating least squares technique has been proposed. In [6], an approach to solve the problem over the smooth Riemannian manifold of rank-$\eta$ matrices has been proposed.

In this paper, we propose a matrix completion algorithm referred to as conjugate gradient based localization in smooth Riemannian manifold (CGL-SRM) for pursuing efficiency in the reconstruction of the location information of sensor nodes. In CGL-SRM, we formulate the matrix completion problem in (2) into the unconstrained optimization in smooth Riemannian manifold. On the designed manifold, we employ a nonlinear conjugate gradient algorithm. From the numerical results, we show that the proposed CGL-SRM algorithm recovers the distance matrix accurately both in noiseless and noisy conditions.
We summarize notations used in this paper. For a given matrix $A$, $\text{diag}(A)$ is the vector formed by its main diagonal and $\text{Sym}(A)$ is defined by $\text{Sym}(A) = \frac{1}{2}(A + A^T)$. For an orthogonal matrix $Q \in \mathbb{R}^{n \times k}$ with $n > k$, we define its orthogonal complement $Q_{\perp} \in \mathbb{R}^{n \times (n-k)}$ such that $[Q \; Q_{\perp}]$ forms an orthonormal matrix. Given a function $f : Y \to f(Y)$, $\nabla_Y f(Y)$ is the Euclidean gradient of $f(Y)$ with respect to $Y$, i.e., $\nabla_Y f(Y)|_{ij} = \frac{\partial f(Y)}{\partial y_{ij}}$.

II. MATRIX COMPLETION OVER SMOOTH RIEMANNIAN MANIFOLD

In this section, we present the proposed CGL-SRM algorithm to complete the sparse Euclidean distance matrix. In order to transform the matrix completion problem in (2) into an unconstrained optimization problem on the manifold, we exploit the smooth manifold structure of the low-rank symmetric PSD matrices.

A. Problem Model

From the definition of pairwise distance $d_{ij}^2 = \|x_i - x_j\|^2 = x_i^T x_i + x_j^T x_j - 2x_i^T x_j$, we have

$$D = \kappa(XX^T),$$

where $\kappa(XX^T) = 1\text{diag}(XX^T)^T + \text{diag}(XX^T)1^T - 2XX^T$. The next lemma follows immediately from (3).

**Lemma II.1.** When $n$ sensor nodes are distributed in $k$-dimensional Euclidean space and $n \geq k$, then $\text{rank}(D) \leq k + 2$.

From this lemma, (2) can be rewritten as

$$\begin{align*}
\min_{D \in \mathbb{R}^{n \times n}} & \quad \frac{1}{2}||P_E(\widetilde{D}) - D_{\text{obs}}||_F^2, \\
\text{s.t.} & \quad \text{rank}(\widetilde{D}) \leq k + 2.
\end{align*}$$

**B. Optimization over Riemannian Manifold**

Since we assume that sensor nodes are randomly distributed in $k$-dimensional Euclidean space (i.e., entries of $X$ are i.i.d.), $\text{rank}(X) = k$ is ensured almost surely. Thus, we can strengthen the constraint set $\mathcal{Y}$ to $\mathcal{Y} = \{XX^T : X \in \mathbb{R}^{n \times k}, \text{rank}(X) = k\}$ and the modified problem is

$$\min_{Y \in \mathcal{Y}} \frac{1}{2}||P_E(\kappa(Y)) - D_{\text{obs}}||_F^2.$$  

In the sequel, we denote $f(Y) = \frac{1}{2}||P_E(\kappa(Y)) - D_{\text{obs}}||_F^2$. Now, if we define $\mathcal{S} = \{U \in \mathbb{R}^{n \times k} : U^T U = I_k\}$ and $\mathcal{L} = \{\text{eye}([\lambda_1 \ldots \lambda_k]^T) : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0\}$, then an element $Y \in \mathcal{Y}$ can be expressed as $Y = QAQ^T$ where $Q \in \mathcal{S}$ and $A \in \mathcal{L}$. That is,

$$\mathcal{Y} = \{QAQ^T : Q \in \mathcal{S}, A \in \mathcal{L}\}.$$  

It can be shown that $\mathcal{Y}$ is a smooth Riemannian manifold [8, Proposition 1.1].

Since the tools we use in this work is based on the differential geometry, we briefly introduce key ingredients to describe the proposed CGL-SRM algorithm. First, the tangent space of $\mathcal{Y}$ is characterized in the following lemma.

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1. $\mathcal{S}$ is an orthogonal Stiefel manifold embedded in $\mathbb{R}^{n \times k}$ [7].
Lemma II.2. For the manifold $\bar{Y}$ defined in (7), the tangent space $T_Y\bar{Y}$ at $Y$ is
\[
T_Y\bar{Y} = \begin{cases} [Q \ Q_{\perp}] & \begin{bmatrix} B & C^T \ C & 0 \ \end{bmatrix} [Q^T] \\ \end{cases}
= \begin{bmatrix} QBQ^T + QQ_p^T + Q_pQ^T : B \in \mathbb{R}^{k \times k}, \\ B^T = B, Q_p = \frac{1}{2} C, \text{ and } C \in \mathbb{R}^{(n-k) \times k} \end{bmatrix}.
\]

A metric on the tangent space $T_Y\bar{Y}$ is defined as the matrix inner product between two tangent vectors $\beta_1, \beta_2 \in T_Y\bar{Y}$, that is, $\langle \beta_1, \beta_2 \rangle = tr(\beta_1^T \beta_2)$.

Lemma II.3. For a given matrix $A$, orthogonal projection $P_{T_Y\bar{Y}}(A)$ of $A$ on the tangent space $T_Y\bar{Y}$ is
\[
P_{T_Y\bar{Y}}(A) = P_{\mathcal{Q}}\text{Sym}(A) + \text{Sym}(A)P_{\mathcal{Q}} - P_{\mathcal{Q}}\text{Sym}(A)P_{\mathcal{Q}},
\]
where $P_{\mathcal{Q}} = QQ^T$.

To express the concept of moving in the direction of a tangent space yet staying on the manifold, we employ an operation called retraction. Note that the retraction operation is a mapping from $T_Y\bar{Y}$ to $\bar{Y}$ that preserves the gradient at $Y$ [7].

Definition II.4. The retraction $R_Y(\beta)$ of a vector $\beta \in T_Y\bar{Y}$ onto $\bar{Y}$ is defined as
\[
R_Y(\beta) = \arg \min_{Z \in \bar{Y}} \| Y + \beta - Z \|_F. \tag{8}
\]

Since $R_Y(\beta)$ is on $\bar{Y}$, we focus only on the symmetric part. For a given matrix $A$, if we take the eigenvalue decomposition of its symmetric part $\text{Sym}(A) = P\Sigma P^T$, then $W_k(A)$ is defined as
\[
W_k(A) = P\Sigma_k^+ P^T, \tag{9}
\]
where $\Sigma_k^+ = \text{eye} \left( \begin{bmatrix} \sigma_1^+ & \cdots & \sigma_k^+ \\ 0 & \cdots & 0 \end{bmatrix}^T \right)$. After some computations, one can show that
\[
R_Y(\beta) = W_k(Y + \beta). \tag{10}
\]

C. Conjugate Gradient based Localization

In the proposed CGL-SRM method, we apply the CG method to solve (6). Note that the update formula of the conventional CG algorithm is
\[
Y_{i+1} = Y_i + \alpha_i P_i, \tag{11}
\]
where $\alpha_i$ is the stepsize and $P_i$ is the conjugate direction. In our problem, since the conjugate direction $P_i$ should be on the tangent space, we use a retraction operator to map this into the element in the manifold. Using (10), we have
\[
Y_{i+1} = W_k(Y_i + \alpha_i P_i). \tag{12}
\]
By this operation, we can ensure that the updated point $Y_{i+1}$ is on $\bar{Y}$.

Now what remains is to compute $P_i$ and $\alpha_i$. First, in order to obtain $P_i$, we need to obtain the Riemannian gradient $\nabla f(Y)$ of $f(Y)$. $\nabla f(Y)$ is expressed as
\[
\nabla f(Y) = P_{T_Y\bar{Y}}(\nabla Y f(Y)).
\]
In the traditional CG algorithm, the conjugate direction $P_i$ has the update formula as
\[
P_i = -\nabla f(Y_i) + \beta_i P_{i-1}, \tag{13}
\]

Algorithm 1: The proposed CGL-SRM algorithm
\begin{algorithm}[!ht]
\caption{The proposed CGL-SRM algorithm}
\begin{algorithmic}[1]
\State Input: $D_{obs}, P_E, \tau$: tolerance, $\sigma \in (0, 1)$
\State $T$: number of iterations
\State Initialize: $Y_1 \in \bar{Y}$, $i = 1$, tangent vector $P_0$
\While {$i \leq T$}
\State $R_i = P_E(\kappa(\bar{Y}_i)) - D_{obs}$
\State $\nabla f(Y_i) = 2\text{eye}(\text{Sym}(R_i))1 - 2R_i$
\State $\nabla f(Y_i) = P_{T_Y\bar{Y}}(\nabla Y f(Y_i))$
\State $H_i = \nabla f(Y_i) - P_{T_Y\bar{Y}}(\nabla f(Y_{i-1}))$
\State $\beta_i = \frac{1}{\|H_i\|_F^2} \|\nabla f(Y_i)\|_F$
\State $P_i = -\nabla f(Y_i) + \beta_i P_{T_Y\bar{Y}}(\nabla f(Y_{i-1}))$
\State $\alpha_i > 0$ such that $\nabla f(Y_i) - (\nabla f(Y_i) - P_{T_Y\bar{Y}}(\nabla f(Y_{i-1}))) \leq -\sigma \alpha_i < \nabla f(Y_i), P_i >$
\State $\bar{Y}_{i+1} = W_k(Y_i + \alpha_i P_i)$
\State $D_{i+1} = \kappa(Y_{i+1})$
\If {$\|P_E(D_{i+1}) - D_{obs}\|_F \leq \tau$} \break \EndIf
\State $i = i + 1$
\EndWhile
\State Obtain the eigenvalue decomposition $\bar{Y} = QAQ^T$
\State $Z = QA^{1/2}$
\State Output: $Z$
\end{algorithmic}
\end{algorithm}

where $\beta_i$ is the conjugate update parameter. However, $\nabla f(Y_i)$ and $P_{i-1}$ lie in two different vector spaces $T_Y\bar{Y}$ and $T_{Y_{i-1}}\bar{Y}$, so in order to add two vectors together, we need to project $P_{i-1}$ onto $T_Y\bar{Y}$. Thus, the update formula of CGL-SRM is
\[
P_i = -\nabla f(Y_i) + \beta_i P_{T_Y\bar{Y}}(\nabla f(Y_{i-1})). \tag{14}
\]

For the stepsize $\alpha_i$, we apply Armijo’s rule. In this rule, a finite number of trials are performed to search a stepsize $\alpha_i$ that loosely satisfies $\min f(R_Y(\alpha_i P_i))$. To be specific, we find the stepsize $\alpha_i$ satisfying
\[
f(R_Y(\alpha_i P_i)) \leq f(Y_i) + \sigma \alpha_i < \nabla f(Y_i), P_i >,
\]
where $\sigma$ is a given constant $0 < \sigma < 1$.

III. SIMULATION AND DISCUSSIONS

A. Simulation Setup

In this section, we compare the performance of the proposed CGL-SRM with TNN-ADMM [9], LMaFit [5], LRGeomCG [6], ADMiRA [10], and SDP solver for the nuclear norm minimization (SDPT3) [11]. Let $D$ be the recovered Euclidean distance matrix. As a metric to measure the recovery performance, we use the normalized root mean square error (RMSE):
\[
RMSE = \frac{\|\hat{D} - D\|_F}{\|D\|_F}. \tag{15}
\]
In our experiment, we generate 200 random sensor nodes whose positions are randomly drawn from the uniform distribution on $[0, 1]$. For each experiment, we run more than 1000 trials to compute the average RMSE.
B. Recovery Performance

In Fig. 2, we present the matrix completion performance in the noiseless scenario as a function of the sampling ratio. Fig. 2 shows that CGL-SRM accurately reconstructs the distance matrix even with very small number of observations and also outperforms conventional algorithms. In the noisy scenario, the observed pairwise distances are modelled as $D_{obs} = P_D(D + N)$. In Fig. 3, we set the average of the sampling ratio to 0.5. As shown in Fig. 3, we observe that the proposed CGL-SRM algorithm exhibits excellent matrix completion performance even in the low SNR regime and also outperforms conventional algorithms.

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