On the Gaussian Z-Interference channel

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Abstract

The optimality of the Han and Kobayashi achievable region (with Gaussian signaling) remains an open problem for Gaussian interference channels. In this paper we focus on the Gaussian Z-interference channel. We first show that using correlated (over time) Gaussian signals does not improve on the Han and Kobayashi achievable rate region. Secondly we compute the slope of the Han and Kobayashi achievable region with Gaussian signaling around the Sato's corner point.

I. Introduction

Gaussian interference channel is one of the most basic multiuser settings whose capacity region is as yet undetermined. The best known-achievable region for a two-receiver interference channel is due to Han and Kobayashi [1]. Recently it has been shown that there are two-receiver interference channels with discrete alphabets where the Han and Kobayashi region is strictly inside the capacity region [2]. However for the Gaussian setting, the optimality of the Han and Kobayashi region (with Gaussian auxiliaries) remains an open challenge.

In the discrete memoryless setting, it was shown that a 2-letter extension (coding in blocks of two symbols) of the Han and Kobayashi scheme strictly outperforms the single-letter (traditional) scheme. There has been some attempts, for instance, see [3], at using correlated Gaussians to improve on the Han and Kobayashi scheme for the interference channel.

Remark 1. It is worth mentioning that the authors in [3] claim that the rates they achieve outperform state-of-the-art coding schemes. This claim fails to hold if a comparison is made with the rates of the Han and Kobayashi scheme with Gaussian signals and the use of a time-sharing variable, Q, to do power control. A scheme employing the concept of *noisebergs* has been shown by one of the authors [4] to strictly improve the rate region with power control. The sub-optimality of the region without power control can also be shown by using perturbations along Hermite polynomials [5].

In this paper, the first result is a proof that correlated Gaussian signaling does not improve on the traditional single-letter scheme for the Gaussian interference channel. The second result concerns evaluation of the slope of the Han and Kobayashi region with Gaussian signaling around the corner point known as Sato's corner point.

A. Preliminaries

An interference channel is a model for communication where two point-to-point communications occur over a shared medium causing interference. The particular channel model that we study in this paper is called the Gaussian Z-interference setting, and the channel is described by:

$$\begin{aligned} Y_1 &= X_1 + Z_1 \\ Y_2 &= X_2 + aX_1 + Z_2. \end{aligned} \tag{1}$$

Here Z_1 and Z_2 are Gaussian variables, each distributed as $\mathcal{N}(0,1)$ and independent of X_1 and X_2. We further assume power constraints P_1, P_2 on X_1, X_2 and that $a \in (0,1)$. (Note that if $a = 0$ or $a \geq 1$, then the capacity region is fully determined; hence this regime is the only open case.)

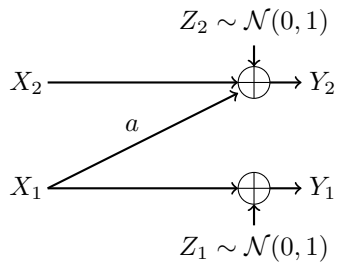

Fig. 1: Gaussian Z interference channel

The capacity region for this setting is defined in the usual sense (see [6] for details and background work).

The Han and Kobayashi achievable region [1] for the interference channel can be found in Section 6.5 of [6]. However when the interference is one-sided (as in the Z-channel), the achievable region simplifies to the following.

Theorem 1 (Han and Kobayashi Region)**.** *The union of rate pairs* (R_1, R_2) *satisfying the constraints*

$$\begin{aligned} R_1 &\leq I(X_1; Y_1|Q) \\ R_2 &\leq I(X_2; Y_2|U_1, Q) \\ R_1 + R_2 &\leq I(U_1, X_2; Y_2|Q) + I(X_1; Y_1|U_1, Q) \end{aligned}$$

over distributions $p_Q(q)p_{U_1,X_1|Q}(u_1, x_1|q)p_{X_2|Q}(x_2|q)$ *is achievable for a (discrete memoryless)* Z*-interference channel. To achieve this region, it suffices to consider* $|Q| \leq 3$*.*

The above region will also yield an achievable region in the Gaussian Z-interference channel defined by (7). Further, for every $Q = q$, if $X_1 = U_1 + V_1$, where U_1 and V_1 are zero-mean independent Gaussian random variables, and X_2 is also an independent Gaussian random variable, then we call such a region as *Han and Kobayashi region with Gaussian signaling*.

Theorem 2 (Han and Kobayashi region with Gaussian signaling)**.** *The union of rate pairs* (R_1, R_2) *satisfying the constraints*

$$\begin{aligned} R_1 &\leq \mathrm{E}_Q\left(\frac{1}{2}\log\left(1 + P_{1Q}\right)\right) \\ R_2 &\leq \mathrm{E}_Q\left(\frac{1}{2}\log\left(1 + \frac{P_{2Q}}{1 + a^2\alpha_Q P_{1Q}}\right)\right) \\ R_1 + R_2 &\leq \mathrm{E}_Q\left(\frac{1}{2}\log\left(1 + \alpha_Q P_{1Q}\right) + \frac{1}{2}\log\left(1 + \frac{P_{2Q} + a^2(1-\alpha_Q)P_{1Q}}{1 + a^2\alpha_Q P_{1Q}}\right)\right) \end{aligned}$$

over $\alpha_Q \in [0, 1], P_{1Q}, P_{2Q} \geq 0$ *satisfying* $\mathrm{E}_Q\left(P_{1Q}\right) \leq P_1$ *and* $\mathrm{E}_Q\left(P_{2Q}\right) \leq P_2$ *is achievable.*

By Bunt's extension of Caratheodory's theorem, it suffices to consider $|Q| \leq 5$. The region described by Theorem 2 will be referred to as $\mathcal{R}_{HK}$.

By computing the Han and Kobayashi region with Gaussian signaling of the multi-letter extension of the Gaussian interference channel, the following region is achievable.

Theorem 3 (k-letter Han and Kobayashi region with Gaussian signaling)**.** *The union of rate pairs* (R_1, R_2) *satisfying the constraints*

$$\begin{aligned} R_1 &\leq \frac{1}{k}\,\mathrm{E}_Q\left(\frac{1}{2}\log|I + K_{1Q}|\right) \\ R_2 &\leq \frac{1}{k}\,\mathrm{E}_Q\left(\frac{1}{2}\log\frac{|I + a^2K_{vQ} + K_{2Q}|}{|I + a^2K_{vQ}|}\right) \\ R_1 + R_2 &\leq \frac{1}{k}\,\mathrm{E}_Q\left(\frac{1}{2}\log|I + K_{vQ}| + \frac{1}{2}\log\frac{|I + a^2K_{1Q} + K_{2Q}|}{|I + a^2K_{vQ}|}\right) \end{aligned}$$

over $K_{1q}, K_{2q} \succeq 0, K_{vq} \preceq K_{1q}$ *satisfying* $\frac{1}{k}\,\mathrm{E}_Q(tr(K_{1Q})) \leq P_1$ *and* $\frac{1}{k}\,\mathrm{E}_Q(tr(K_{2Q})) \leq P_2$ *is achievable.*

As before it suffices to consider $|Q| \leq 5$. The $\succeq$ relation denotes the positive semi-definite partial order among real symmetric matrices; while $|A|$ and $tr(A)$ denotes the determinant and trace of matrix A, respectively. The region described by Theorem 3 will be referred to as $\mathcal{R}_{HK}^{(k)}$. Clearly $\mathcal{R}_{HK} \subseteq \mathcal{R}_{HK}^{(k)} \subseteq \mathcal{C}$, the capacity region.

1) Known results about the capacity region: In this section we summarize the previously known results about the capacity region of the Gaussian Z-interference channel.

- (i) It is known (from [7], [8], [9]) that the rate-pair $R_1 = \frac{1}{2}\log(1 + P_1)$ and $R_2 = \frac{1}{2}\log\left(1 + \frac{P_2}{1+a^2P_1}\right)$ is a Pareto-optimal point on the boundary of the capacity region. Further it is also known that the above point maximizes the rate-sum $R_1 + R_2$. Since $C_1 = \frac{1}{2}\log(1 + P_1)$ is the maximum achievable rate to receiver Y_1, the capacity region contains a line-segment that starts at $(C_1, 0)$ and ends at $(C_1, \frac{1}{2}\log\left(1 + \frac{P_2}{1+a^2P_1}\right))$. We call this extremal point (corner point) as *Sato-point*.
  The outer bounds in [9] and [10] shows that the *Sato-point* also maximizes $\beta R_2 + R_1$ for any $\beta \leq \frac{1+a^2P_1}{a^2(1+P_1)}$. This provides an outer bound to the slope of the capacity region around the Sato's corner point.
- (ii) It is known (from [7], [11], [12]) that the rate-pair $R_1 = \frac{1}{2}\log\left(1 + \frac{a^2P_1}{1+P_2}\right)$ and $R_2 = C_2 = \frac{1}{2}\log(1 + P_2)$, is another Pareto-optimal point on the boundary of the capacity region of the Gaussian Z interference channel. Hence the capacity region contains a line-segment that starts at $(0, C_2)$ and ends at $\left(\frac{1}{2}\log\left(1 + \frac{a^2P_1}{1+P_2}\right), C_2\right)$. This recently established extremal point is the second corner point of the capacity region of the Z Gaussian interference channel. The outer bound to the capacity region of the interference channel does not yield any finite β such that the maximum of $\beta R_2 + R_1$ over

achievable rate pairs passes through this corner point. Two of the authors computed the slope of Han and Kobayashi region with Gaussian signaling around this new corner point [13].

B. Summary of our results

In this article we establish the following results.

Theorem 4. $\mathcal{R}_{HK}^{(k)} = \mathcal{R}_{HK}, \forall k \geq 1.$

We show that the k-letter extension of the Han and Kobayashi region with Gaussian signaling does not improve on the single-letter scheme.

Theorem 5. *The largest value of β such that the maximum of $\beta R_2 + R_1$ (with $(R_1, R_2) \in \mathcal{R}_{HK}$) occurs at the Sato point is given by*

$$\beta_{cr}^{HK,s} = \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \beta^*\right\}$$

where β^* is the unique positive solution of $\gamma(\beta^*) = 0$, where

$$\gamma(\beta) := \beta\left(\log\left(1+\frac{P_2}{1+a^2P_1}\right) - \frac{(1-a^2)P_2}{(1+a^2P_1)(1+a^2P_1+P_2)}\right) + \log\left(1-\frac{a^2P_2(1+P_1)}{(1+a^2P_1)(1+a^2P_1+P_2)}\beta\right).$$

II. Correlated Gaussians do not improve the region

In this section we establish Theorem 4. Towards this end we make the following observations: both $\mathcal{R}_{HK}^{(k)}$ and $\mathcal{R}_{HK}$ are convex regions, hence they can be characterized by the intersection of supporting hyperplanes. Further the hyperplane $R_1 + R_2$ (to the capacity region) passes through the Sato-point, which is present in both $\mathcal{R}_{HK}^{(k)}$ and $\mathcal{R}_{HK}$. Hence Theorem 4 is equivalent to showing that for every $\beta > 1$,

$$\max_{(R_1,R_2)\in\mathcal{R}_{HK}^{(k)}} \beta R_2 + R_1 = \max_{(R_1,R_2)\in\mathcal{R}_{HK}} \beta R_2 + R_1.$$

The above condition can be re-expressed in terms of matrices as

$$\begin{aligned}
&\max_{\substack{K_{1q},K_{2q}\succeq 0, K_{vq}\preceq K_{1q}:\\ \frac{1}{k}\mathrm{E}_Q(tr(K_{1Q}))\leq P_1, \frac{1}{k}\mathrm{E}_Q(tr(K_{2Q}))\leq P_2}} \mathrm{E}_Q\Big(\frac{1}{2k}\log|K_{2Q}+a^2K_{1Q}+I| + \frac{(\beta-1)}{2k}\log|K_{2Q}+a^2K_{vQ}+I| \\
&\quad + \frac{1}{2k}\log|K_{vQ}+I| - \frac{\beta}{2k}\log|a^2K_{vQ}+I|\Big) \qquad (2)\\
&= \max_{\substack{P_{1q},P_{2q}\succeq 0, \alpha_q\in[0,1]:\\ \mathrm{E}_Q(tr(P_{1Q}))\leq P_1, \mathrm{E}_Q(tr(P_{2Q}))\leq P_2}} \mathrm{E}_Q\Big(\frac{1}{2}\log|P_{2Q}+a^2P_{1Q}+1| + \frac{(\beta-1)}{2}\log|P_{2Q}+a^2\alpha_QP_{1Q}+1| \\
&\quad + \frac{1}{2}\log|\alpha_QP_{1Q}+1| - \frac{\beta}{2}\log|a^2\alpha_QP_{1Q}+1|\Big).
\end{aligned}$$

The key to establishing (2) is the following Theorem.

Theorem 6. *The maximum of the expression*

$$\frac{1}{2}\log|K_2+a^2K_v+a^2K_u+I| + \frac{(\beta-1)}{2}\log|K_2+a^2K_v+I| + \frac{1}{2}\log|K_v+I| - \frac{\beta}{2}\log|a^2K_v+I|$$

where the $k\times k$ Hermitian matrices satisfy the constraints: $K_2, K_u, K_v \succeq 0, tr(K_u+K_v) \leq P_1, tr(K_2) \leq P_2$ can be attained by restricting K_2, K_u, K_v to be diagonal matrices.

Remark 2. Identify $K_{uq} := K_{1q} - K_{vq}$. It is immediate that equation (2) will follow from Theorem 6 by the following reasoning: for every $Q = q$, Theorem 6 allows us to replace the matrices inside the expression by diagonal matrices, which then is just sum of terms of the form appearing on the right-hand-side. Note the extra factor $\frac{1}{k}$ on the left hand-side changes this sum into a convex combination and the equality is immediate.

The following notations will be used in the proof:

(i) Given a $k\times k$ matrix A, let $\lambda(A)$ denote the (unordered) set of eigenvalues of A, and let $\lambda^{\downarrow}(A)$ ($\lambda^{\uparrow}(A)$) denote the k-tuple of eigenvalues of A arranged in decreasing (increasing) order respectively.

(ii) Given two vectors v, w, we say $v \gg w$ if v *majorizes* w, i.e. if $v_{[1]} \geq v_{[2]} \cdots \geq v_{[k]}$ is a non-increasing arrangement of v and $w_{[1]} \geq w_{[2]} \cdots \geq w_{[k]}$ is a non-increasing arrangement of w; then

$$\sum_{l=1}^{m} v_{[l]} \geq \sum_{l=1}^{m} w_{[l]}, \quad 1 \leq m \leq k$$

with equality at $m = k$.

Proof of Theorem 6. Suppose we fix the matrices K_2 and K_v satisfying the trace constraint; then we must choose K_u so as to maximize $|K_2 + a^2 K_v + a^2 K_u + I|$ subject to $tr(K_u) \leq P_1 - tr(K_v)$.

If one further fixes the eigenvalues $\lambda(K_u)$ then Fiedler's bound [14] says that

$$|K_2 + a^2 K_v + a^2 K_u + I| \leq \prod_{i=1}^{k} \left(\lambda_i^{\downarrow}(K_2 + a^2 K_v + I) + a^2 \lambda_i^{\uparrow}(K_u) \right),$$

and clearly equality is achieved if the matrices $K_2 + a^2 K_v + I$ and K_u share the same eigenvectors with eigenvalues $\lambda_i^{\downarrow}(K_2 + a^2 K_v + I)$ and $\lambda_i^{\uparrow}(K_u)$, respectively. Hence we seek to maximize

$$\prod_{i=1}^{k} \left(\lambda_i^{\downarrow}(K_2 + a^2 K_v + I) + a^2 \lambda_i^{\uparrow}(K_u) \right),$$

subject to $\sum_{i=1}^{k} \lambda_i^{\uparrow}(K_u) \leq P_1 - tr(K_v)$ and $\lambda_i^{\uparrow}(K_u) \geq 0$. The optimal choice of this problem is the *water-filling solution*. Denote the optimal choice as K_u^w; then it is immediate that

$$\lambda_i^{\downarrow}(K_2 + a^2 K_v + I + a^2 K_u) = \lambda_i^{\downarrow}(K_2 + a^2 K_v + I) + a^2 \lambda_i^{\uparrow}(K_u^w), \quad i = 1, .., k.$$

Let K_2^* be a diagonal matrix with entries ordered as $\lambda_i^{\downarrow}(K_2)$, and K_v^* be a diagonal matrix with entries ordered as $\lambda_i^{\uparrow}(K_v)$. Applying Fiedler's bound [14] we see that

$$|K_2 + a^2 K_v + I| \leq \prod_{i=1}^{k} \left(\lambda_i^{\downarrow}(K_2) + \lambda_i^{\uparrow}(K_v) + 1 \right) = |K_2^* + a^2 K_v^* + I|. \tag{3}$$

We now invoke the celebrated Lidskii-Wielandt inequality (see (2.6) in survey article [15]) that establishes the majorization,

$$\lambda^{\downarrow}(K_2^* + a^2 K_v^* + I) = \lambda^{\downarrow}(K_2) + \lambda^{\uparrow}(a^2 K_v + I) \ll \lambda^{\downarrow}(K_2 + a^2 K_v + I).$$

Let $K_u^{*,w}$ denotes the diagonal water filling matrix corresponding to $K_2^* + a^2 K_v^* + I$. Lemma 1 implies that

$$\lambda^{\downarrow}(K_2^* + a^2 K_v^* + I + K_u^{*,w}) \ll \lambda^{\downarrow}(K_2 + a^2 K_v + I + K_u^w).$$

Lemma 2 yields

$$|K_2^* + a^2 K_v^* + I + K_u^{*,w}| \geq |K_2 + a^2 K_v + I + K_u^w|. \tag{4}$$

Since $|K_v + I| = \prod_{i=1}^{k}(1 + \lambda_i(K_v))$ and $|a^2 K_v + I| = \prod_{i=1}^{k}(1 + a^2 \lambda_i(K_v))$, we see that for a fixed choice of $\lambda(K_v)$ and $\lambda(K_2)$, the diagonal matrices K_2^*, K_v^*, and $K_u^{*,w}$ maximize the expression (term-by-term) in Theorem 6. Varying over the choices of $\lambda(K_v)$ and $\lambda(K_2)$ that satisfies the trace constraint establishes the theorem. □

A. Lemmas regarding majorization and its applications

The following Lemma must be well-known but we cannot find an immediate reference, so we establish it below.

Lemma 1. *[Waterfilling preserves majorization] Let v, u be vectors such that $v \ll u$. Let v' and u' denote the vectors obtained after water-filling operation with a quantity of water $W > 0$. Then $v' \ll u'$.*

Proof. W.l.o.g. let v and u be arranged in decreasing order. After waterfilling operation note that v' and u' is also in decreasing order; and further they satisfy

$$v_i' = \begin{cases} v_i & 1 \leq i \leq m \\ c & m+1 \leq i \leq k \end{cases},$$

$$u_i' = \begin{cases} u_i & 1 \leq i \leq n \\ c_1 & n+1 \leq i \leq k \end{cases}.$$

Further, $v_m \geq c \geq v_{m+1}$, $u_n \geq c_1 \geq u_{n+1}$, and $W = \sum_{i=m+1}^{k}(c - v_i) = \sum_{i=n+1}^{k}(c_1 - u_i)$.

We divide the proof into two cases: $m < n$ and $m \geq n$.

Case 1: $m < n$. Note that

$$\begin{aligned}\sum_{i=1}^{k} v_i + W &= \sum_{i=1}^{k} u_i + W \\ \implies \sum_{i=1}^{m} v_i + (k-m)c &= \sum_{i=1}^{n} u_i + (k-n)c_1 \geq \sum_{i=1}^{m} u_i + (k-m)c_1 \\ &\geq \sum_{i=1}^{m} v_i + (k-m)c_1.\end{aligned}$$

The first inequality is due to $u_i \geq c_1, m+1 \leq i \leq n$, and the second inequality is from $v \preceq u$. Hence $c \geq c_1$. Thus to establish that $v' \ll u'$, it suffices to show that

$$\sum_{i=1}^{l} v'_i \leq \sum_{i=1}^{l} u'_i, \quad m+1 \leq l \leq n,$$

as the rest of the choices of l are immediate from $v \preceq u$, $c \geq c_1$, and that $\sum_{i=1}^{k} v'_i = \sum_{i=1}^{k} u'_i$.

We establish it by contradiction. Suppose l_o is the first index in $[m+1 : n]$ such that

$$\sum_{i=1}^{m} v_i + \sum_{i=m+1}^{l_o} c = \sum_{i=1}^{l_o} v'_i > \sum_{i=1}^{l_o} u'_i = \sum_{i=1}^{l_o} u_i.$$

Hence it must be that $c > u_{l_o} = u'_{l_o}$, and since u'_i is decreasing, $c \geq u'_i, \ \forall i \geq l_o$. This would imply that

$$\sum_{i=1}^{k} v'_i = \sum_{i=1}^{l_o} v'_i + \sum_{i=l_o+1}^{k} c > \sum_{i=1}^{l_o} u'_i + \sum_{i=l_o+1}^{k} u'_i = \sum_{i=1}^{k} u'_i,$$

a contradiction. This completes the proof in this case.

Case 2: $m \geq n$. Note that

$$W = \sum_{i=m+1}^{k} (c - v_i) = \sum_{i=n+1}^{k} (c_1 - u_i) \geq \sum_{i=m+1}^{k} (c_1 - u_i) \geq \sum_{i=m+1}^{k} (c_1 - v_i),$$

where the first inequality is due to $c - u_i \geq 0, \quad n+1 \leq i \leq m$, and the second inequality is from $v \ll u$, (tail of v has larger partial sum than tail of u). Thus $c \geq c_1$, as before. Similar to previous case, to establish that $v' \ll u'$, it suffices to show that

$$\sum_{i=1}^{l} v'_i \leq \sum_{i=1}^{l} u'_i, \quad n+1 \leq l \leq m,$$

as the rest of the choices of l are immediate from $v \preceq u$, $c \geq c_1$, and that $\sum_{i=1}^{k} v'_i = \sum_{i=1}^{k} u'_i$. The proof follows again by contradiction. Suppose l_o is the first index in $[n+1 : m]$ such that

$$\sum_{i=1}^{l_o} v_i = \sum_{i=1}^{l_o} v'_i > \sum_{i=1}^{l_0} u'_i = \sum_{i=1}^{n} u_i + \sum_{i=n+1}^{l_0} c_1.$$

Hence it must be that $v_{l_0} > c_1$, and since v'_i is decreasing, $v'_i \geq c_1, \ \forall i \geq l_0$. This would imply that

$$\sum_{i=1}^{k} v'_i = \sum_{i=1}^{l_o} v'_i + \sum_{i=l_o+1}^{k} v'_i > \sum_{i=1}^{l_o} u'_i + \sum_{i=l_o+1}^{k} c_1 = \sum_{i=1}^{k} u'_i,$$

a contradiction. This completes the proof of the lemma. □

Lemma 2. *(see A.1.d, page 166 in [16]) Let $A, B \succ 0$ and $\lambda(A) \ll \lambda(B)$. Then $\prod_{i=1}^{k} \lambda_i(A) = |A| \leq |B| = \prod_{i=1}^{k} \lambda_i(B)$.*

It basically follows from the concavity of $\log(\cdot)$ and the Hardy-Littlewood-Polya majorization inequality.

III. Slope at Sato's corner point

A. Han and Kobayashi region with Gaussian signaling

For $\beta \geq 1$, the maximum of the weighted sum-rate $R_1 + \beta R_2$ of the Han and Kobayashi region (with Gaussian signaling) for a Z-interference channel can be computed as

$$\max_{\substack{P_{1q},P_{2q}\succeq 0,\alpha_q\in[0,1]:\\ \mathrm{E}_Q(tr(P_{1Q}))\leq P_1,\mathrm{E}_Q(tr(P_{2Q}))\leq P_2}} \mathrm{E}_Q\left(\frac{1}{2}\log|P_{2Q}+a^2P_{1Q}+1| + \frac{(\beta-1)}{2}\log|P_{2Q}+a^2\alpha_Q P_{1Q}+1|\right.$$
$$\left. + \frac{1}{2}\log|\alpha_Q P_{1Q}+1| - \frac{\beta}{2}\log|a^2\alpha_Q P_{1Q}+1|\right). \tag{5}$$

We know that when $\beta = 1$, the maximum sum-rate (for the Han and Kobayashi region as well as the capacity region) is given by

$$\frac{1}{2}\log(1+P_1) + \frac{1}{2}\log\left(1+\frac{P_2}{a^2P_1+1}\right)$$

and is achieved at the rate pair $R_1 = \frac{1}{2}\log(1+P_1)$, and $R_2 = \frac{1}{2}\log\left(1+\frac{P_2}{a^2P_1+1}\right)$, which is referred to as Sato's corner point.

We define

$$\beta_{cr}^{HK,s} = \sup\left\{\beta : \max_{(R_1,R_2)\in\mathcal{R}_{HK}} R_1+\beta R_2 = \frac{1}{2}\log(1+P_1) + \beta\frac{1}{2}\log\left(1+\frac{P_2}{a^2P_1+1}\right)\right\},$$

the largest β such that the line $R_1 + \beta R_2$ passing through Sato's corner point is a supporting hyperplane to $\mathcal{R}_{HK}$.

Using the concave envelope interpretation (see [17]) for E_Q we see that for any $\beta \geq 1$ the value of $\max_{(R_1,R_2)\in\mathcal{R}_{HK}} R_1+\beta R_2$ is the upper concave envelope of $f_\beta(Q_1,Q_2)$ evaluated at (P_1,P_2), where f_β is defined by

$$f_\beta(Q_1,Q_2) := \frac{1}{2}\log(1+a^2Q_1+Q_2) + \max_{\alpha\in[0,1]}\left\{\frac{\beta}{2}\log\frac{1+a^2\alpha Q_1+Q_2}{1+a^2\alpha Q_1} + \frac{1}{2}\log\frac{1+\alpha Q_1}{1+a^2\alpha Q_1+Q_2}\right\}$$

By taking derivative with respect to α, the optimal $\alpha = \alpha^*$ satisfies:

$$\beta = \frac{1-a^2+Q_2}{a^2Q_2}\left(a^2+\frac{1-a^2}{1+\alpha^*Q_1}\right)$$

We define the regions

$$\begin{aligned}
\mathcal{R}_1 &= \left\{(Q_1,Q_2) : \beta \geq \frac{1-a^2+Q_2}{a^2Q_2}\right\}\\
\mathcal{R}_2 &= \left\{(Q_1,Q_2) : \beta \leq \frac{(1-a^2+Q_2)(1+a^2Q_1)}{a^2Q_2(1+Q_1)}\right\}\\
\mathcal{R}_3 &= \left\{(Q_1,Q_2) : \frac{(1-a^2+Q_2)(1+a^2Q_1)}{a^2Q_2(1+Q_1)} < \beta < \frac{1-a^2+Q_2}{a^2Q_2}\right\}
\end{aligned} \tag{6}$$

where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ correspond to the cases $\alpha^* = 0$, $\alpha^* = 1$ and $0 < \alpha^* < 1$ respectively.

This gives an explicit expression for f_β,

$$f_\beta(Q_1,Q_2) = \begin{cases}
\frac{1}{2}\log(1+a^2Q_1+Q_2) + \frac{\beta-1}{2}\log(1+Q_2) & (Q_1,Q_2)\in\mathcal{R}_1\\
\frac{\beta}{2}\log(1+\frac{Q_2}{1+a^2Q_1}) + \frac{1}{2}\log(1+Q_1) & (Q_1,Q_2)\in\mathcal{R}_2\\
\frac{1}{2}\log\frac{1+a^2Q_1+Q_2}{a^2Q_2} + \frac{\beta-1}{2}\log(\beta-1) - \frac{\beta}{2}\log\beta + \frac{\beta}{2}\log\frac{1-a^2+Q_2}{1-a^2} + \frac{1}{2}\log(1-a^2) & (Q_1,Q_2)\in\mathcal{R}_3
\end{cases}$$

Note that the hyperplane $R_1 + \beta R_2$ passes through Sato's corner point if and only if $(P_1,P_2)\in\mathcal{R}_2$ and $\mathcal{C}f_\beta(P_1,P_2) = f_\beta(P_1,P_2)$. From Lemma 11 in Appendix, this is equivalent to requiring that $(P_1,P_2)\in\mathcal{R}_2$ and $g_\beta(Q_1,Q_2)$ attains global maximum at (P_1,P_2), where g_β is defined by

$$g_\beta(Q_1,Q_2) := f_\beta(Q_1,Q_2) - \frac{1}{2}\left(\frac{a^2\beta}{1+a^2P_1+P_2} - \frac{a^2\beta}{1+a^2P_1} + \frac{1}{1+P_1}\right)Q_1 - \frac{1}{2}\left(\frac{\beta}{1+a^2P_1+P_2}\right)Q_2.$$

Thus to establish Theorem 5, we need to show that $g_\beta(Q_1,Q_2)$ attains a global maximum at (P_1,P_2) if and only if $\beta \leq \beta_{cr}^{HK,s}$ where

$$\beta_{cr}^{HK,s} = \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \beta^*\right\}$$

where β^* is the solution to $h(\beta^*) = 0$, with

$$h(\beta) := \beta\left\{\log(1+\frac{P_2}{1+a^2P_1}) - \frac{(1-a^2)P_2}{(1+a^2P_1)(1+a^2P_1+P_2)}\right\} + \log\left\{1 - \frac{a^2P_2(1+P_1)}{(1+a^2P_1)(1+a^2P_1+P_2)}\beta\right\}.$$

The proof of Theorem 5 is completed by analysing the local behaviour of g_β and isolating the local maxima.

1) Interior analysis:

Lemma 3.

$$\beta_{cr}^{HK,s} \leq \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\right\}$$

Proof. This is the condition that says $(P_1, P_2) \in \mathcal{R}_2$ and it is a local maximum. $\beta_{cr}^{HK,s} \leq \frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}$ since $(P_1, P_2) \in \mathcal{R}_2$; else the optimizing α is not 1 (see (6)). The second condition, $\beta_{cr}^{HK,s} \leq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$ follows from one of the second order conditions for (P_1, P_2) being a local maximum, namely $\det \mathcal{H} f_\beta(P_1, P_2) \geq 0$ (see Hessian calculation in Appendix). □

Lemma 4. *There is no local maximum of g_β in the interior of $\mathcal{R}_1$ for any $\beta \leq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$.*

Proof. Since g_β is concave in $\mathcal{R}_1$, there is at most one local maximum in the interior of $\mathcal{R}_1$. The first order condition yields

$$\frac{a^2}{2}\frac{1}{1+a^2Q_1+Q_2} = \frac{1}{2}\left(\frac{a^2\beta}{1+a^2P_1+P_2} - \frac{a^2\beta}{1+a^2P_1} + \frac{1}{1+P_1}\right)$$
$$\frac{1}{2}\frac{1}{1+a^2Q_1+Q_2} + \frac{\beta-1}{2}\frac{1}{1+Q_2} = \frac{1}{2}\left(\frac{\beta}{1+a^2P_1+P_2}\right)$$

Solving for Q_2 gives

$$Q_2 = \frac{(1+a^2P_1)(1+P_1)}{1+P_1+\frac{1-\frac{1}{a^2}}{\beta-1}}$$

But in $\mathcal{R}_1$ we have $\beta \geq \frac{1-a^2+Q_2}{a^2Q_2}$. Substituting for Q_2 yields

$$\beta \geq \frac{1+a^2P_1}{a^4(1+P_1)}.$$

But we also have $\beta \leq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$, implying $a^2 \geq 1$. This gives a contradiction. □

Lemma 5. *There are at most 2 local maxima of g_β in the interior of $\mathcal{R}_2$. The value of g_β at both points is bounded from above by $g_\beta(P_1, P_2)$, when $\beta \leq \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$.*

Proof. The first order condition for local-maximum yields

$$\frac{a^2\beta}{1+a^2Q_1+Q_2} - \frac{a^2\beta}{1+a^2Q_1} + \frac{1}{1+Q_1} = \frac{a^2\beta}{1+a^2P_1+P_2} - \frac{a^2\beta}{1+a^2P_1} + \frac{1}{1+P_1}$$
$$\frac{\beta}{1+a^2Q_1+Q_2} = \frac{\beta}{1+a^2P_1+P_2}$$

The solutions are

$$Q_1 = P_1 \text{ or } \frac{\frac{1}{a^2}-1}{\frac{\beta}{k}-1} - 1$$
$$Q_2 = P_2 + a^2(P_1 - Q_1)$$

where $k := \frac{1+a^2P_1}{a^2(1+P_1)} \geq 1$. If $k \geq \beta$, there is only one solution (in $\mathbb{R}_+^2$) at (P_1, P_2), so assume that $k < \beta$.

If (Q_1, Q_2) is any solution to above, then

$$\begin{aligned} g_\beta(Q_1, Q_2) &= f_\beta(Q_1, Q_2) - \frac{\beta}{2}\frac{a^2Q_1+Q_2}{1+a^2Q_1+Q_2} + \frac{\beta}{2}\frac{a^2Q_1}{1+a^2Q_1} - \frac{1}{2}\frac{Q_1}{1+Q_1} \\ &= f_\beta(Q_1, Q_2) + \frac{\beta}{2}\frac{1}{1+a^2Q_1+Q_2} - \frac{\beta}{2}\frac{1}{1+a^2Q_1} + \frac{1}{2}\frac{1}{1+Q_1} - \frac{1}{2} \\ &= \frac{\beta}{2}\varphi\left(1+a^2Q_1+Q_2\right) - \frac{\beta}{2}\varphi\left(1+a^2Q_1\right) + \frac{1}{2}\varphi\left(1+Q_1\right) - \frac{1}{2} \\ &= \frac{\beta}{2}\varphi\left(1+a^2P_1+P_2\right) - \frac{\beta}{2}\varphi\left(1+a^2Q_1\right) + \frac{1}{2}\varphi\left(1+Q_1\right) - \frac{1}{2} \end{aligned}$$

where $\varphi(x) := \log x + \frac{1}{x}$.

Now let (Q_1, Q_2) to be the solution other than (P_1, P_2). Then,

$$\begin{aligned} g_\beta(P_1,P_2) - g_\beta(Q_1,Q_2) &= \frac{\beta}{2}\left(\varphi\left(1+a^2Q_1\right) - \varphi\left(1+a^2P_1\right)\right) - \frac{1}{2}\left(\varphi\left(1+Q_1\right) - \varphi\left(1+P_1\right)\right) \\ &= \frac{\beta}{2}\left(\varphi\left((1-a^2)\frac{\beta}{\beta-k}\right) - \varphi\left((1-a^2)\frac{k}{k-1}\right)\right) \\ &\quad - \frac{1}{2}\left(\varphi\left(\frac{1-a^2}{a^2}\frac{k}{\beta-k}\right) - \varphi\left(\frac{1-a^2}{a^2}\frac{1}{k-1}\right)\right) \end{aligned}$$

Differentiating with respect to β and simplifying gives

$$\partial_\beta\left(g_\beta(P_1,P_2) - g_\beta(Q_1,Q_2)\right) = \frac{1}{2}\left[\log\left(1+\frac{k^2-\beta}{k(\beta-k)}\right) - \frac{k^2-\beta}{k(\beta-k)}\right] \le 0$$

since $\beta \le \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2 = k^2$ and $\forall x \ge 0, \log(1+x) \le x$. So

$$g_\beta(P_1,P_2) - g_\beta(Q_1,Q_2) \ge g_{\beta=k^2}(P_1,P_2) - g_{\beta=k^2}(Q_1,Q_2) = 0$$

and hence $g_\beta(P_1,P_2) \ge g_\beta(Q_1,Q_2)$. □

Lemma 6. *There is no local maximum of g_β in the interior of $\mathcal{R}_3$ when $\beta < \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$.*

Proof. The first order condition for g_β having a local maximum yields

$$\begin{aligned} \frac{a^2}{1+a^2Q_1+Q_2} &= \frac{a^2\beta}{1+a^2P_1+P_2} - \frac{a^2\beta}{1+a^2P_1} + \frac{1}{1+P_1} \\ \frac{1}{1+a^2Q_1+Q_2} - \frac{1}{Q_2} + \frac{\beta}{1-a^2+Q_2} &= \frac{\beta}{1+a^2P_1+P_2}. \end{aligned}$$

By substituting the first equation into the second one, and then writing $\beta = \frac{1-a^2+Q_2}{a^2Q_2}\theta$, where $\theta \in \left(\frac{1+a^2Q_1}{1+Q_1}, 1\right)$, we get

$$Q_2 = a^2(1+P_1)$$

From the second order condition (see Appendix) $\det \mathcal{H}f_\beta(Q_1,Q_2) \ge 0$, or equivalently,

$$\begin{aligned} \beta &\ge \left(\frac{1-a^2+Q_2}{Q_2}\right)^2 \\ &= \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2 \end{aligned}$$

which contradicts with that $\beta < \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$. □

Thus the interior analysis yields that the value of $g_\beta(Q_1,Q_2)$ at any interior local maximum is upper bounded by that of $g_\beta(P_1,P_2)$ and $(P_1,P_2) \in \mathcal{R}_2$ if and only if

$$\beta \le \min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}, \left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\right\}.$$

The necessity comes from Lemmas 3, while the sufficiency comes from Lemmas 4, 5, and 6.

2) Boundary analysis: The remaining cases are the boundaries $Q_1 = 0$ and $Q_2 = 0$. In this part, we first establish that $g_\beta(P_1,P_2) \ge g_\beta(Q_1,Q_2)$ for (Q_1,Q_2) on the boundaries if and only if β is smaller than or equal to the upper bound in Lemma 3 and β^* in Theorem 5. Then in Lemma 10 we reduce the minimum of three terms to that of two of them. This gives the critical β in Theorem 5.

Lemma 7. *On the boundary $Q_1 = 0$ we have that*

$$g_\beta(P_1,P_2) \le \max_{Q_2 \ge 0} g_\beta(0,Q_2)$$

if and only if $\beta \le \frac{\log(1+P_1)+\frac{1}{1+P_1}-1}{\log(1+a^2P_1)+\frac{1}{1+a^2P_1}-1}$.

Proof. When $Q_1 = 0$, we have

$$\begin{aligned} f_\beta(Q_1,Q_2) &= \frac{\beta}{2}\log(1+Q_2), \\ g_\beta(Q_1,Q_2) &= \frac{\beta}{2}\log(1+Q_2) - \frac{\beta}{2}\frac{1}{1+a^2P_1+P_2}Q_2. \end{aligned}$$

Note that $g_\beta(0, Q_2)$ is concave in Q_2, and maximized at $Q_2 = a^2P_1 + P_2$. Since $(P_1, P_2) \in \mathcal{R}_2$,

$$\begin{aligned}
&\min_{Q_2 \geq 0} \left(g_\beta(P_1, P_2) - g_\beta(0, Q_2)\right) \\
&= g_\beta(P_1, P_2) - g_\beta(0, a^2P_1 + P_2) \\
&= -\frac{\beta}{2}\left(\log(1 + a^2P_1) + \frac{1}{1 + a^2P_1} - 1\right) + \frac{1}{2}\left(\log(1 + P_1) + \frac{1}{1 + P_1} - 1\right)
\end{aligned}$$

≥ 0 if and only if $\beta \leq \frac{\log(1+P_1)+\frac{1}{1+P_1}-1}{\log(1+a^2P_1)+\frac{1}{1+a^2P_1}-1}$. □

Lemma 8.

$$\frac{\log(1 + P_1) + \frac{1}{1+P_1} - 1}{\log(1 + a^2P_1) + \frac{1}{1+a^2P_1} - 1} \geq \left(\frac{1 + a^2P_1}{a^2(1 + P_1)}\right)^2$$

and hence, by Lemma 3 and Lemma 7, $g_\beta(P_1, P_2) \geq g_\beta(Q_1, Q_2)$ on the boundary $Q_1 = 0$.

Proof. This is equivalent to

$$a^4\varphi\left(P_1\right) - \varphi\left(a^2P_1\right) \geq 0$$

where $\varphi\left(x\right) = (1 + x)^2 \log(1 + x) - (1 + x)x$. Let

$$\psi(x) = a^4\varphi(x) - \varphi(a^2x).$$

Note that

$$\begin{aligned}
\varphi'\left(x\right) &= 2(1 + x)\log(1 + x) - x \\
\varphi''\left(x\right) &= 2\log(1 + x) + 1
\end{aligned}$$

Hence

$$\begin{aligned}
\partial_x\left(\psi(x)\right) &= a^4\varphi'\left(x\right) - a^2\varphi'\left(a^2x\right) \\
\partial_x^2\left(\psi(x)\right) &= a^4 \cdot 2\log\frac{1 + x}{1 + a^2x} \geq 0.
\end{aligned}$$

So $\psi(x)$ is convex when $x \geq 0$ and $\psi'(0) = 0$, implying ψ is convex and increasing on $x \geq 0$. Thus $\psi(P_1) \geq \psi(0) = 0$. □

Lemma 9. *On the boundary $Q_2 = 0$ we have that*

$$g_\beta(P_1, P_2) \leq \max_{Q_1 \geq 0} g_\beta(Q_1, 0)$$

if and only if $\beta \leq \beta^*$, where β^* as in Theorem 5.

Proof. When $Q_2 = 0$, we have

$$\begin{aligned}
f_\beta(Q_1, Q_2) &= \frac{1}{2}\log(1 + Q_1) \\
g_\beta(Q_1, Q_2) &= \frac{1}{2}\log(1 + Q_1) - \frac{1}{2}\left(\frac{a^2\beta}{1 + a^2P_1 + P_2} - \frac{a^2\beta}{1 + a^2P_1} + \frac{1}{1 + P_1}\right)Q_1
\end{aligned}$$

$g_\beta(Q_1, 0)$ is concave in Q_1, maximized when

$$\begin{aligned}
\frac{1}{1 + Q_1} &= \frac{a^2\beta}{1 + a^2P_1 + P_2} - \frac{a^2\beta}{1 + a^2P_1} + \frac{1}{1 + P_1} \\
&\in [0, 1] \text{ since } \beta \leq \frac{(1 - a^2 + P_2)(1 + a^2P_1)}{a^2P_2(1 + P_1)}
\end{aligned}$$

That is, there is always a maximizing $Q_1 \geq 0$. Note that, after some manipulations, we can express

$$\begin{aligned}
&\min_{Q_1 \geq 0} \left(g_\beta(P_1, P_2) - g_\beta(Q_1, 0)\right) \\
&= \frac{1}{2}\left[\beta\left\{\log(1 + \frac{P_2}{1 + a^2P_1}) - \frac{(1 - a^2)P_2}{(1 + a^2P_1)(1 + a^2P_1 + P_2)}\right\} + \log\left\{1 - \frac{a^2P_2(1 + P_1)}{(1 + a^2P_1)(1 + a^2P_1 + P_2)}\beta\right\}\right] \\
&= \frac{1}{2}h(\beta)
\end{aligned}$$

which is concave in β, equals 0 when $\beta = 0$, the derivative with respect to β is non-negative at $\beta = 0$. Here $h(\cdot)$ as in Theorem 5. Hence $g_\beta(P_1, P_2) \leq \max_{Q_1 \geq 0} g_\beta(Q_1, 0)$ if and only if $\beta \leq \beta^*$. □

Thus combining the interior and boundary analysis we see that $\beta_{cr}^{HK,s}$ is the minimum of three quantities, two of them given by Lemma 3 and one given by Lemma 9. The proof is completed by showing that one of the three quantities is redundant.

Lemma 10. *The following holds:*

$$\min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)},\beta^*\right\}=\min\left\{\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)},\left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2,\beta^*\right\}$$

where β^* as in Theorem 5.

Proof. It suffices to show that, if $\frac{(1-a^2+P_2)(1+a^2P_1)}{a^2P_2(1+P_1)}\geq\left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2$, or equivalently $P_2\leq a^2(1+P_1)$, then $\left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\geq\beta^*$. That is, $\gamma\left(\left(\frac{1+a^2P_1}{a^2(1+P_1)}\right)^2\right)\leq 0$, where $\gamma(\cdot)$ is defined in Theorem 5.

Write $P_2=(1+a^2P_1)\theta$ and $k:=\frac{1+a^2P_1}{a^2(1+P_1)}$. Then $\theta\leq\frac{1}{k}$ and $k\geq 1$. We would like to show $h(k^2)\leq 0$, that is,

$$\begin{aligned}
& k^2\log(1+\theta)-k(k-1)\frac{\theta}{\theta+1}+\log(1-\frac{k\theta}{1+\theta})\leq 0\\
\Leftrightarrow\qquad & k^2\left(\log(1+\theta)+\frac{1}{1+\theta}-1\right)+\left(\frac{k\theta}{1+\theta}+\log(1-\frac{k\theta}{1+\theta})\right)\leq 0
\end{aligned}$$

The derivative of left hand side with respect to θ is equal to

$$\begin{aligned}
& k^2\frac{\theta}{(1+\theta)^2}+\frac{k}{(1+\theta)^2}\frac{k\theta}{k\theta-(1+\theta)}\\
&=\frac{k^2\theta}{(1+\theta)^2}\left(1+\frac{1}{k\theta-(1+\theta)}\right)\\
&=\frac{k^2\theta^2}{(1+\theta)^2}\frac{1-k}{1+\theta-k\theta}\\
&\leq 0
\end{aligned}$$

So $k^2\left(\log(1+\theta)+\frac{1}{1+\theta}-1\right)+\left(\frac{k\theta}{1+\theta}+\log(1-\frac{k\theta}{1+\theta})\right)$ is decreasing in θ, and $=0$ when $\theta=0$. We are done. □

This completes the proof of Theorem 5.

B. Outerbound to the slope at Sato's point

The outer bound the the capacity region by Sato [18] and Kramer [10] (see also [11] from which this form is taken) states that any achievable rate-pair (R_1,R_2) must satisfy

$$e^{-2R_1}\leq\left(\frac{1+a^2P_1+P_2}{a^2}\right)e^{-2R_1}-\frac{1}{a^2}+1.$$

A simple calculus argument shows that the largest β such that the maximum of $R_1+\beta R_2$ occurs at Sato's corner point is given by

$$\beta_{cr}^{OB,s}:=\frac{1+a^2P_1}{a^2(1+P_1)}.$$

Remark 3. Note that the inner bound for the slope $\beta_{cr}^{HK,s}$ coincides with the outer bound $\beta_{cr}^{OB,s}$ in the limit $P_2\to\infty$.

CONCLUSION

In this paper we show that correlated Gaussians (over time) do not improve on the single-letter Han and Kobayashi achievable region (with Gaussian signaling). We also computed the slope of the Han and Kobayashi region around the Sato's corner point.

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APPENDIX

A. Calculation via the Noiseberg region

One of the authors proposed [4] that the Han and Kobayashi region (with Gaussian signaling) for the Z-interference channel is equivalent to the *noiseberg region*. This is based on the equivalence between the Gaussian Z-interference channel and the Gaussian degraded interference channel, demonstrated in [7]. The Gaussian degraded interference channel is described by

$$\begin{aligned} Y_1' &= X_1' + Z_1' \\ Y_2' &= X_2' + Y_1' + Z_2' \end{aligned} \tag{7}$$

where Z_1' and Z_2' are Gaussian variables distributed as $\mathcal{N}(0,1)$ and $\mathcal{N}(0,N_2)$, respectively, and independent of X_1' and X_2'. In addition we assume power constraints P_1', P_2' on X_1', X_2'. The equivalence between the Z and the degraded Gaussian interference channels holds if their three parameters are related by $P_1' = P_1$, $P_2' = \frac{P_2}{a^2}$ and $N_2 = \frac{1-a^2}{a^2}$. Applying the notation in Theorem 2, the noiseberg scheme (a particular choice of parameters) corresponds to $Q = 2$ with: $\mathrm{P}(Q=1) = 1-\lambda$, $\mathrm{P}(Q=2) = \lambda$, $\alpha_2 = 1$, $P_{22} = 0$. This leads to the constraints

$$\begin{aligned} R_1 &\le \frac{1-\lambda}{2}\log(1+P_{11}) + \frac{\lambda}{2}\log(1+P_{12}) \\ R_2 &\le \frac{1-\lambda}{2}\log\left(1+\frac{P_{21}}{1+a^2\alpha_1 P_{11}}\right) \\ R_1+R_2 &\le (1-\lambda)\left(\frac{1}{2}\log\left(1+\alpha_1 P_{11}\right) + \frac{1}{2}\log\left(1+\frac{P_{21}+a^2(1-\alpha_1)P_{11}}{1+a^2\alpha_1 P_{11}}\right)\right) + \frac{\lambda}{2}\log(1+P_{12}). \end{aligned}$$

We have the constraints $P_{21} = \frac{P_2}{1-\lambda}, P_{12} = \frac{P_1-(1-\lambda)P_{11}}{\lambda}$. Therefore there are three free variables in the above expression: $P_{11}, \alpha_1, \lambda$. Therefore the maximum weighted sum-rate of the noiseberg region is given by

$$\begin{aligned} R_1+\beta R_2 = \max_{\alpha_1, P_{11}, \lambda} &(1-\lambda)\left(\frac{1}{2}\log\left(1+\alpha_1 P_{11}\right) + \frac{1}{2}\log\left(1+\frac{a^2(1-\alpha_1)P_{11}}{1+a^2\alpha_1 P_{11}+\frac{P_2}{1-\lambda}}\right)\right) + \frac{\lambda}{2}\log\left(1+\frac{P_1-(1-\lambda)P_{11}}{\lambda}\right) \\ &+ \beta\frac{(1-\lambda)}{2}\log\left(1+\frac{\frac{P_2}{1-\lambda}}{1+a^2\alpha_1 P_{11}}\right). \end{aligned}$$

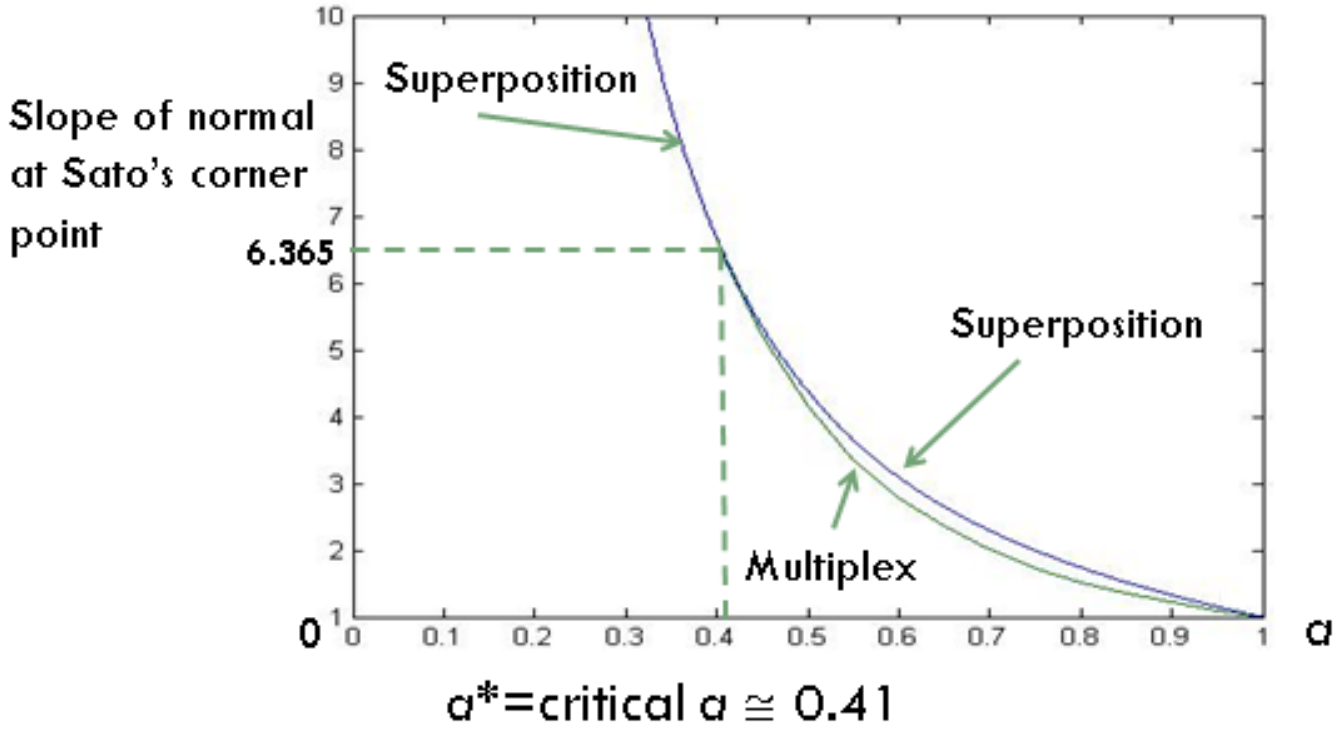

Fig. 2: Slope of normal at Sato's point for $P_1 = P_2 = 1$.

The noiseberg scheme applied to the degraded interference channel uses two parameters, namely the multiplex band λ and the noiseberg height h. The rates R_1 and R_2 associated with this scheme are given by

$$R_1 = \frac{1-\lambda}{2}\left(\log(1+\frac{P_{1A}}{1-\lambda}) + \log(1+\frac{\max(0,h-N_2)}{1+N_2+\frac{P_{1A}}{1-\lambda}+\frac{P_2'}{1-\lambda}})\right) + \frac{\lambda}{2}\log(1+\frac{P_{1B}}{\lambda})$$

$$R_2 = \frac{1-\lambda}{2}\log(1+\frac{\frac{P_2'}{1-\lambda}}{1+N_2+\frac{P_{1A}}{1-\lambda}})$$

where the partial powers P_{1A} and P_{2A} are given by

$$P_{1A} = (1-\lambda)\left[P_1 - \lambda\min(h,N_2) - \max(0,h-N_2)\right] - \lambda P_2'$$
$$P_{2A} = \lambda\left[P_1 + P_2' + (1-\lambda)\min(h,N_2)\right].$$

To simplify notation we restricted the primed power to P_2'. The parameters λ and h are defined in a certain admissible region (see details in [4]). The corner point that corresponds to Sato's point is $(R_1 = \frac{1}{2}\log(1+P_1), R_2 = \frac{1}{2}\log(1+\frac{P_2'}{1+P_1+N_2}))$. To find the contour of the Gaussian Han and Kobayashi region at this point we evaluate the gradient of $R_1(h,\lambda) + \beta R2_(h,\lambda)$, for λ close to zero and $0 \le h \le N_2$. Equating the gradient to zero for the outermost rate contour we find

$$\frac{dR_1/d\lambda}{dR_2/d\lambda} = \frac{dR_1/dh}{dR_2/dh} = -\beta. \tag{8}$$

Note that β is the slope of the normal to the rate contour of the achievable region.

As calculated in [19], the first of this derivative ratios can be expressed as

$$\frac{dR_1/d\lambda}{dR_2/d\lambda} = \frac{\frac{P_2'+h}{1+P_1} - \log\left(1+\frac{P_2'+h}{1+P_1}\right)}{\frac{P_2'}{1+P_1+N_2} - \log\left(1+\frac{P_2'}{1+P_1+N_2}\right) + \frac{hP_2'}{(1+P_1+P_2')(1+P_1+P_2'+N_2)}} \tag{9}$$

The second ratio of derivatives is

$$\frac{dR_1/dh}{dR_2/dh} = \frac{(P_2'+h)(1+P_1+N_2)(1+P_1+P_2'+N_2)}{P_2'(1+P_1)(1+P_1+P_2'+h)} \tag{10}$$

Example plots of these derivative ratios are given in [19] for $P_1 = 1$, $P_2' = 4$ and $N_2 = 3$, i. e., $P_1 = 1$, $P_2 = 1$ and $a = 0.5$. To find the point of intersection we can equate these two expressions and do some algebraic manipulation. Equivalently, we can take the derivative of Eq. (9) with respect to h and equate it to zero. We then get the following equation.

$$(1 + P_1 + P_2' + N_2)G\left(\frac{P_2'}{1 + P_1 + N_2}\right) - N_2 = (1 + P_1 + P_2' + h)G\left(\frac{P_2' + h}{1 + P_1}\right), \tag{11}$$

where the function $G(x)$ is defined as $(\log(1 + x))/x$.

Solving this equation numerically or graphically for h we obtain a side product of this approach which is the optimal initial height of the noiseberg, as the rate point departs from Sato's point, if $h \leq N_2$. Let this optimal value be denoted by h^*. Then we can use Eqs. (9) or (10) with $h = h^*$, and then Eq. (8) to get the normal slope β at Sato's point. For example, if we take the case $P_1 = 1$, $P_2' = 4$ and $N_2 = 3$, equivalent to $P_1 = P_2 = 1$ and $a = 0.5$, we get $h^* = 1.5415$, and $\beta = 4.1333$. If the h solution to Eq. (11) turns out to be greater than N_2, then there is no need for multiplexing with a noiseberg band, and the optimal strategy to exit the Sato's point is pure superposition, that is moving along the path with $\lambda = 0$ and h varying from N_2 to $N_2 + P_2'$ in the admissible (λ, h) parameter region. In this case the normal slope at Sato's point is easily found to be

$$\beta = \frac{(P_2' + N_2)(1 + P_1 + N_2)}{P_2'(1 + P_1)}. \tag{12}$$

In Fig. 2 we plot the slope of the normal to the achievable region at Sato's point for a in the interval $[0, 1]$ with $P_1 = P_2 = 1$. We show two curves which correspond to the optimal multiplexing strategy (bottom curve) and to the pure superposition scheme (top curve). Note that for values of a below approximately 0.41, the two curves coincide, indicating that for these values of the interference gain, there is no advantage in multiplexing (See "To mux and not to mux" in [19]).

Remark 4. To illustrate the need for multiplexing for certain values of the channel parameters, we observe that the linear combination of rates R_1 and R_2 given by $R_1 + \beta R_2 = \frac{1}{2}\log(1 + P_1) + \frac{\beta}{2}\log(1 + \frac{P_2}{1+P_1+N_2})$is not concave in certain regions of the (P_1, P_2) plane and certain values of β. In these cases the best rate combinations happen above the surface of the function, in its concave envelope, and require the multiplexing of two superposition schemes. As an example, in Fig.3 we show a surface plot of this function in the plane (P_1, P_2) with $\beta = 4.1333$ and $N_2 = 3$, for $0 \leq (P_1, P_2) \leq 8$. The shading of the surface indicate the non-concavity.

B. Gradient and Hessian of the function f_β

In $\mathcal{R}_1$,

$$\begin{aligned}
\partial_{Q_1} f_\beta &= \frac{a^2}{2}\frac{1}{1 + a^2Q_1 + Q_2} \\
\partial_{Q_2} f_\beta &= \frac{1}{2}\frac{1}{1 + a^2Q_1 + Q_2} + \frac{\beta - 1}{2}\frac{1}{1 + Q_2} \\
\mathcal{H} f_\beta &= \begin{bmatrix} \frac{-a^4}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} \\ \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-1}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} - \frac{\beta-1}{2}\frac{1}{(1+Q_2)^2} \end{bmatrix}
\end{aligned}$$

In $\mathcal{R}_2$,

$$\begin{aligned}
\partial_{Q_1} f_\beta &= \frac{1}{2}\left(\frac{a^2\beta}{1 + a^2Q_1 + Q_2} - \frac{a^2\beta}{1 + a^2Q_1} + \frac{1}{1 + Q_1}\right) \\
\partial_{Q_2} f_\beta &= \frac{\beta}{2}\frac{1}{1 + a^2Q_1 + Q_2} \\
\mathcal{H} f_\beta &= \begin{bmatrix} \frac{1}{2}\left(\frac{-a^4\beta}{(1+a^2Q_1+Q_2)^2} + \frac{a^4\beta}{(1+a^2Q_1)^2} - \frac{1}{(1+Q_1)^2}\right) & \frac{-a^2\beta}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} \\ \frac{-a^2\beta}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-\beta}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} \end{bmatrix}
\end{aligned}$$

In $\mathcal{R}_3$,

$$\begin{aligned}
\partial_{Q_1} f_\beta &= \frac{a^2}{2}\frac{1}{1 + a^2Q_1 + Q_2} \\
\partial_{Q_2} f_\beta &= \frac{1}{2}\left(\frac{1}{1 + a^2Q_1 + Q_2} - \frac{1}{Q_2} + \frac{\beta}{1 - a^2 + Q_2}\right) \\
\mathcal{H} f_\beta &= \begin{bmatrix} \frac{-a^4}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} \\ \frac{-a^2}{2}\frac{1}{(1+a^2Q_1+Q_2)^2} & \frac{1}{2}\left(\frac{-1}{(1+a^2Q_1+Q_2)^2} + \frac{1}{Q_2^2} - \frac{\beta}{(1-a^2+Q_2)^2}\right) \end{bmatrix}
\end{aligned}$$

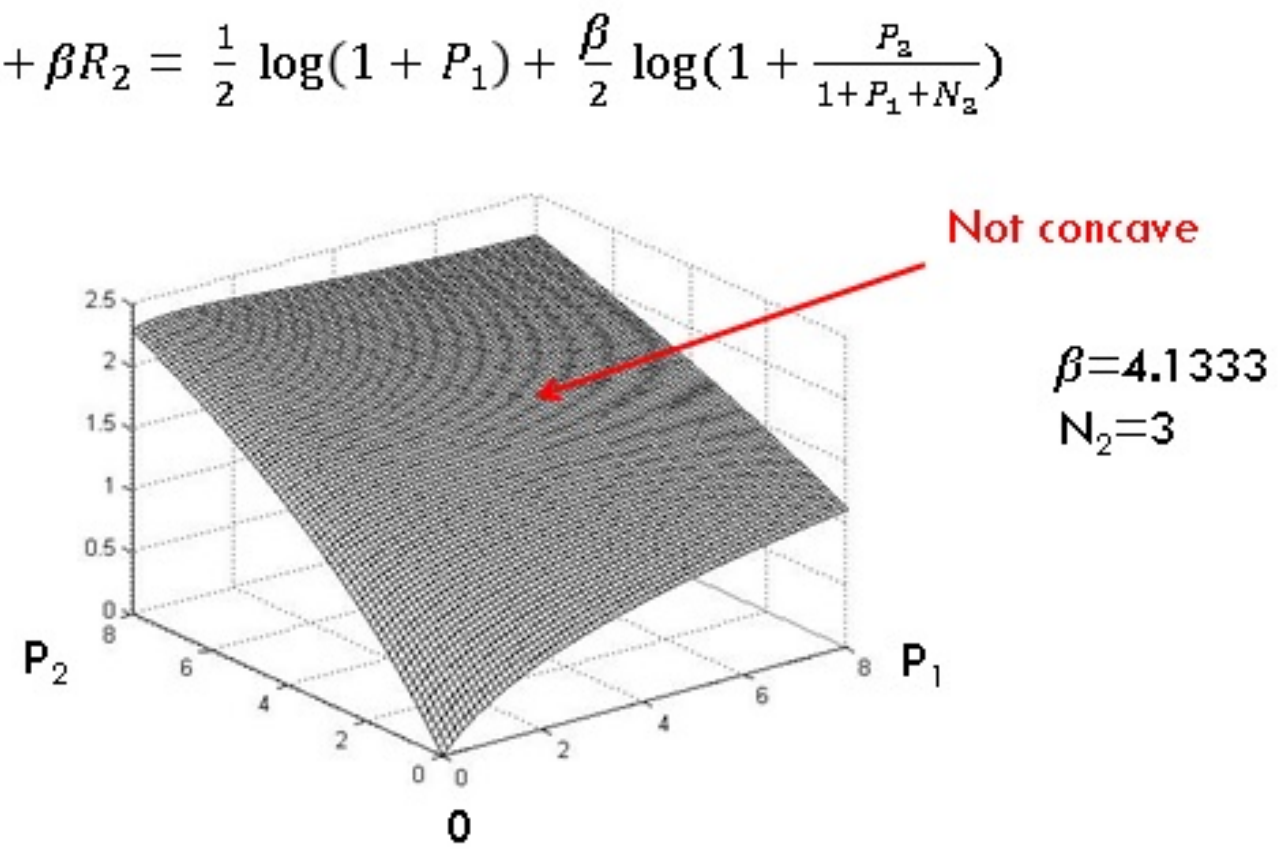

Fig. 3: Surface plot of $R_1 + \beta R_2$ as a function of P_1 and P_2 with $\beta = 4.1333$ and $N_2 = 3$.

By checking the values f_β and ∇f_β at the boundaries, one can see that f_β is continuously differentiable on $\mathbb{R}^2_{>0}$.

Lemma 11. *Let f be a real-valued function differentiable at x. Then $\mathcal{C}f(x) = f(x)$ if and only if $f(\cdot) - \langle \nabla f(x), \cdot \rangle$ attains global maximum at x. Here $\mathcal{C}f$ and ∇f denotes the concave envelope and gradient of f, respectively.*

Proof. It suffices to show that $\mathcal{C}f(x) \le f(x)$ if and only if for all h we have $f(x) \ge f(x+h) - \langle \nabla f(x), h \rangle$. The "if" part is immediate, by taking concave envelope with respect to h and then putting $h = 0$.

For the "only if" part, suppose on the contrary that there is $\epsilon > 0$ and $h \neq 0$ such that

$$f(x) + \epsilon \le f(x+h) - \langle \nabla f(x), h \rangle$$

By differentiability of f at x,

$$|f(x+\zeta) - f(x) - \langle \nabla f(x), \zeta \rangle| \le \frac{\epsilon}{2\|h\|}\|\zeta\|$$

for $\|\zeta\|$ small enough.

Now, for any $\delta \in (0,1)$,

$$\begin{aligned} f(x) &\ge \mathcal{C}f(x) \\ &\ge \delta \cdot \mathcal{C}f(x+h) + (1-\delta) \cdot \mathcal{C}f(x - \frac{\delta}{1-\delta}h) \\ &\ge \delta f(x+h) + (1-\delta) f(x - \frac{\delta}{1-\delta}h) \\ &\ge \delta\epsilon + \delta f(x) + \langle \nabla f(x), \delta h \rangle + (1-\delta) f(x - \frac{\delta}{1-\delta}h) \end{aligned}$$

Rearranging gives

$$\begin{aligned} f(x) &\geq \frac{\delta}{1-\delta}\epsilon + f(x - \frac{\delta}{1-\delta}h) - \left\langle \nabla f(x), -\frac{\delta}{1-\delta}h \right\rangle \\ &\geq \frac{\delta}{1-\delta}\epsilon + f(x) - \frac{\epsilon}{2\|h\|}\left\| -\frac{\delta}{1-\delta}h \right\| \\ &= f(x) + \frac{\epsilon}{2}\frac{\delta}{1-\delta} \end{aligned}$$

for δ small enough. This gives a contradiction.

□