Distributed Mechanism Design for Unicast Transmission

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***Abstract*—The standard (Hurwitz-Reiter) Mechanism Design framework requires that agents broadcast their messages to a central authority that subsequently determines allocation (and tax/subsidies) for each user. We consider a setting where agents can only communicate messages to their neighbors defined by a given communication graph. As a result, allocation and tax functions for each user can only depend on local neighborhood messages. This gives rise to a new, distributed, class of mechanisms. In this paper we propose such a mechanism for the problem of rate allocation over a network with unicast transmission. The proposed mechanism is distributed, it fully implements the optimal allocation in Nash equilibria (ie, there are no extraneous equilibria), is individually rational and weak budget balanced. The message space dimension of the proposed mechanism grows linearly with the number of agents in the network.**

***Index Terms*—mechanism design, rate allocation, decentralized optimization, strategic users, Nash equilibrium**

I. Introduction

In networks with a large number of heterogeneous agents, determining social welfare maximizing allocation of goods and services is significant and complicated because agents have private valuation functions over their allocated goods and they may not be willing to share these functions with a central authority. Moreover, it may not even be possible to transmit these functions due to the communication overhead. Furthermore, the agents may be strategic and wish to maximize their benefit and so they could choose to lie whenever it is profitable. Mechanism design is an appropriate framework to deal with large networks of strategic agents. It is a tool that helps to decentralize the process of reaching a desirable global goal without the need for private information of agents. It has been used in various research areas, such as private/public goods allocations [1]–[3], rate and resource allocations [4]–[7], data security on server farms [8], etc.

The standard (Hurwitz-Reiter) mechanism design framework requires that agents broadcast their messages to a central authority that subsequently determines allocation (and tax/subsidies) for each user. This can cause a communication overhead in large networks even when the message space of the designed mechanism is small. Furthermore, there may be some communication constraints for message transmission. To alleviate this problem, we consider a setting where agents can only communicate messages to their neighbors defined by a given communication graph. As a result, allocation and tax functions for each user can only depend on local neighborhood messages. This gives rise to a new, distributed, class of mechanisms. We call this framework, "Distributed Mechanism Design" (DMD). An alternative viewpoint of DMD is that it tries to augment the area of distributed optimization (see [9], [10]) with the required tools to handle strategic agents. In [11] the authors introduced a distributed mechanism for the problem of Walrasian and Lindhal allocation.

In this paper, we propose a distributed mechanism for the problem of rate allocation over a data communication network with unicast transmission. Non-distributed mechanisms for efficient allocation in the unicast transmission network have been proposed in [12], [13]. Our contributions in this paper are as follows. The proposed mechanism is distributed, it fully implements the optimal allocation in Nash equilibria (ie, there are no extraneous equilibria), and it has message space dimension that grows linearly with the number of agents in the network. Furthermore, individual rationality and weak budget balance are satisfied at NE. In order to achieve all these goals, a number of techniques have been used. For instance, we have utilized an idea similar to the radial allocation [3], [4], [7] to achieve feasibility at NE. Finally, "summary messages" [14, Ch. 4] have been used which enable the message dimensionality of each agent to grow linearly with the size of her neighborhood unlike the mechanism proposed in [11] in which message dimensionality grows with the size of the whole network.

The structure of the paper is as follows. In Section II, the model and problem formulation are discussed. Section III presents the designed distributed mechanism after clarifying all of the assumptions and features of message communication constraints. In Section IV, we characterize the properties of the mechanism. In Section V, two alternative mechanisms are discussed. We conclude in Section VI with some comments on message dimensionality.

II. Model

A pair of transmitter and receiver in the network constitutes an agent. There are N strategic agents in the network that are denoted by $\mathcal{N} = \{1, \ldots, N\}$. Data is communicated via a unicast transmission network. In the unicast transmission, a separate data stream with a specific data rate is

transmitted from each agent's transmitter to its receiver and these data streams pass through network links denoted by $\mathcal{L} = \{1, \ldots, L\}$ and each link $l \in \mathcal{L}$ has capacity c^l. The data stream of agent i passes through links in the set $\mathcal{L}_i \subset \mathcal{L}$ with $|\mathcal{L}_i| = L_i$. Also, the agents using any link l, or in other words the agents for whom their data stream is passing through link l, are denoted by $\mathcal{N}^l$ with $|\mathcal{N}^l| = N^l$. We assume $N^l \geq 2$ that is at least two agents use any link l. This assumption is made so that there is competition between agents for using any link.

The data rate of agent i is x_i and the vector of allocated rates is denoted by $x = (x_1, x_2, ..., x_N)$. Agent i has a valuation function $v_i(x_i)$ that is a function of the data rate she receives. The designer's goal is to determine x to maximize the summation of all of the agents' valuation functions while the network links' capacity constraints are satisfied. Therefore, we can formulize this goal by the following optimization problem,

$$\max_x \sum_{i \in \mathcal{N}} v_i(x_i) \tag{1a}$$

$$\text{s.t.} \quad x_i \geq 0 \quad \forall i \in \mathcal{N} \tag{1b}$$

$$\text{and} \quad \sum_{j \in \mathcal{N}^l} x_j \leq c^l \quad \forall l \in \mathcal{L}. \tag{1c}$$

We impose the following assumptions on the valuation functions. First, $v_i(.) \in \mathcal{V}_0$, where $\mathcal{V}_0$ is the set of strictly concave, twice differentiable, monotonically increasing, $\mathbb{R}_+ \to \mathbb{R}$ functions with continuous second derivatives. Second, for all $i \in \mathcal{N}$, $v_i'(0)$ is finite which implies, due to concavity, that $v_i'(x)$ is finite for all $x \in \mathbb{R}_+$.

Generally, problem (1) can model any scenario of network utility maximization with linear inequality constraints and unicast transmission is just an example of such scenarios.

A. Necessary and Sufficient Optimality Conditions

In order to characterize the solution of problem (1) we use dual variables λ and write the KKT conditions for this problem. Since the valuation functions are concave and all of the constraints of problem (1) are affine, problem (1) is a convex optimization problem and so KKT conditions are necessary and sufficient. These conditions at the optimal point (x^*, λ^*) are

(a) Primal Feasibility: x^* satisfies (1b) and (1c).
(b) Dual Feasibility: $\lambda_l^* \geq 0 \quad \forall l \in \mathcal{L}$
(c) Complimentary Slackness:

$$\lambda_l^*(c^l - \sum_{j \in \mathcal{N}^l} x_j^*) = 0 \quad \forall l \in \mathcal{L} \tag{2}$$

(d) Stationarity:

$$v_i'(x_i^*) = \sum_{l \in \mathcal{L}_i} \lambda_l^* \quad \text{if} \quad x_i^* > 0 \tag{3}$$

$$v_i'(x_i^*) \leq \sum_{l \in \mathcal{L}_i} \lambda_l^* \quad \text{if} \quad x_i^* = 0 \tag{4}$$

III. Distributed Mechanism

The mechanism is a sealed bid auction in which each agent $i \in \mathcal{N}$ has a message space $\mathcal{M}_i$ and allocation and tax functions that are denoted by $\hat{x}_i(.)$ and $\hat{t}_i(.)$ respectively. We can characterize the mechanism completely by specifying the tuple $(\mathcal{M}, (\hat{x}_1(.), \hat{x}_2(.), \ldots, \hat{x}_N(.)), (\hat{t}_1(.), \hat{t}_2(.), \ldots, \hat{t}_N(.)))$, where $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \times \ldots \times \mathcal{M}_N$.

A. Message Network

The agents have access to each other's messages via a network that is modeled by a connected undirected graph. The vertices correspond to the agents and an edge between agent i to j means that agent i can listen to agent j's messages and vice versa. The absence of link between two agents is due to communication constraints between them. We call this network "message network". We emphasize that this is a different network from the one related to problem (1). This network relates to message transmission that enables the decentralized solution of problem (1). In that sense, the message network is relevant even for the more general allocation problems modeled by problem (1).

We further consider an arbitrary spanning tree on the graph of the message network and denote it by $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. In the following we only consider communication over this spanning tree and therefore, we call this the "communication graph" . For all $i \in \mathcal{N}$, $\mathcal{N}(i)$ is defined as the the set of neighbors of agent i in the $\mathcal{G}$ and $n(i, j)$ is the neighboring agent of agent i on the shortest path from i to j. We denote the size of the set $\mathcal{N}(i)$ by $|\mathcal{N}(i)| = N(i)$. For each link $l \in \mathcal{L}_i$, $\mathcal{N}^l(i)$ denotes the set of agents in $\mathcal{N}(i)$ using link l. The size of this set is denoted by $|\mathcal{N}^l(i)| = N^l(i)$. For each agent $i \in \mathcal{N}$, the function $\Phi(i)$ randomly chooses one neighbor of agent i. We can define the set $I_i = \{k \in \mathcal{N}(i) : \Phi(k) = i\}$ based on the function $\Phi(i)$.

For the sake of simplicity of exposition, we first impose the following assumption on the communication graph.

Assumption 1: For each link l, the subgraph consisting of agents $i \in \mathcal{N}^l$ is a connected graph.

This assumption will be relaxed in Section V and two alternative mechanisms will be discussed to avoid imposing this assumption on the graph.

B. Message Components

Agent i announces the message $m_i = (y_i, n_i, q_i, p_i)$, where $y_i \in \mathbb{R}_+$ is a proxy for her demand and $n_i = (n_i^{j,l}, j \in \mathcal{N}(i), l \in \mathcal{L}) \in {\mathbb{R}_+}^{L \times N(i)}$ consists of components $n_i^{j,l}$, each of which is a proxy for the sum of demands of the agents k on link l with $n(i, k) = j$. Further, the message $q_i = (q_i^{j,l}, j \in I_i, l \in \mathcal{L}) \in {\mathbb{R}_+}^{L \times |I_i|}$ is a vector of components $q_i^{j,l}$, each of which is a proxy for the demand of neighboring agent $j \in I_i$. This message has been considered to be used wherever something needs not to be determined by agent i and yet it needs the value of y_i at NE. Finally, the message $p_i = (p_i^l, l \in$

$\mathcal{L}_i) \in {\mathbb{R}_+}^{L_i}$ is the price that agent i thinks every agent using each link $l \in \mathcal{L}_i$ should pay. We define y_i^l as

$$y_i^l = \begin{cases} y_i & \text{if} \quad l \in \mathcal{L}_i \\ 0 & \text{oth.} \end{cases} \tag{5}$$

C. Allocation Functions

In order to have primal feasibility for the optimality conditions, we must define the allocation function in a way that the allocations $\hat{x}_i(m)$ are feasible at Nash equilibria. With this goal in mind, we utilize an idea similar to the radial allocation [7]. The allocation function is defined as

$$\hat{x}_i(m) = r_i y_i \tag{6}$$

where r_i is the radial allocation factor and is defined as

$$r_i = \min_{l \in \mathcal{L}} r_i^l \tag{7}$$

where r_i^l is

$$r_i^l = \begin{cases} \frac{c^l}{f_i^l} & \text{if} \quad f_i^l > 0 \\ \infty & \text{if} \quad f_i^l = 0 \end{cases} \tag{8}$$

and f_i^l is

$$f_i^l = q_{\Phi(i)}^{i,l} + \sum_{j \in \mathcal{N}(i)} (y_j^l + \sum_{k \in \mathcal{N}(j), k \neq i} n_j^{k,l}). \tag{9}$$

The quantity f_i^l is defined in this way to act as a proxy for the sum of demands of agents on link l at NE.

D. Tax Functions

The tax functions are $\hat{t}_i(m) = \sum_{l \in \mathcal{L}} \hat{t}_i^l(m)$ and for each component $\hat{t}_i^l(m)$ we have two cases. For $l \in \mathcal{L}_i$ we have

$$\hat{t}_i^l(m) = \bar{p}_{-i}^l \hat{x}_i(m) + \sum_{j \in \mathcal{N}(i)} (n_i^{j,l} - y_j^l - \sum_{k \in \mathcal{N}(j), k \neq i} n_j^{k,l})^2 + \sum_{j \in I_i} (q_i^{j,l} - y_j^l)^2 + (p_i^l - \bar{p}_{-i}^l)^2 + (p_i^l - \bar{p}_{-i}^l)\bar{p}_{-i}^l (c^l - r_i f_i^l)^2 \tag{10}$$

and $\bar{p}_{-i}^l$ is defined as

$$\bar{p}_{-i}^l = \frac{1}{N^l(i)} \sum_{j \in \mathcal{N}^l(i)} p_j^l. \tag{11}$$

Second, for $l \notin \mathcal{L}_i$,

$$\hat{t}_i^l(m) = \sum_{j \in \mathcal{N}(i)} (n_i^{j,l} - y_j^l - \sum_{k \in \mathcal{N}(j), k \neq i} n_j^{k,l})^2 + \sum_{j \in I_i} (q_i^{j,l} - y_j^l)^2. \tag{12}$$

This mechanism induces a game $\mathfrak{G}_s = (\mathcal{N}, \mathcal{M}_1 \times \ldots \times \mathcal{M}_N, (\hat{u}_1, \ldots, \hat{u}_N))$ where the utility functions are $\hat{u}_i(m) = v_i(\hat{x}_i(m)) - \hat{t}_i(m)$.

IV. Mechanism Properties

Let's denote the set of all Nash equilibria of the game $\mathfrak{G}$ by $\tilde{\mathcal{M}}$ and each NE is denoted by $\tilde{m} = (\tilde{m}_1, ..., \tilde{m}_N)$ and for each $\tilde{m}_i$ we have $\tilde{m}_i = (\tilde{y}_i, \tilde{n}_i, \tilde{q}_i, \tilde{p}_i)$.

Theorem 1: (Full Implementation, Individual Rationality and Weak Budget Balance) For all $\tilde{m} \in \tilde{\mathcal{M}}$ allocation vector $\hat{x}(\tilde{m})$ is efficient; i.e. $\hat{x}(\tilde{m}) = x^*$ where x^* is the solution of problem (1). Furthermore, individual rationality is satisfied for all agents at all NE. Also, we have weak budget balance at NE which means $\sum_{i \in \mathcal{N}} \hat{t}_i \geq 0$.

According to Theorem 1, the allocation vector is unique for all of the Nash equilibria of the game $\mathfrak{G}$, due to uniqueness of x^*.

In order to prove Theorem 1, we first provide a number of lemmas.

Lemma 2: (Concavity) The function $\hat{u}_i(m_i, m_{-i})$ is twice differentiable and strictly concave w.r.t. m_i.

Proof: It is obvious from the definition of the function $\hat{u}_i(m_i, m_{-i})$ and assumptions on the valuation functions that $\hat{u}_i(m_i, m_{-i})$ is twice differentiable w.r.t. m_i. To prove concavity, we calculate the Hessian matrix, H, of $\hat{u}_i(m_i, m_{-i})$ w.r.t. m_i and show that it is a negative definite matrix. As it was defined,

$$\hat{u}_i(m_i, m_{-i}) = v_i(r_i y_i) - \hat{t}_i(y_i, n_i, q_i, p_i, m_{-i}).$$

Since the cross derivatives of $\hat{u}_i(m_i, m_{-i})$ w.r.t. different components of m_i are zero, all of the non-diagonal elements of H are zero. Hence, we calculate the diagonal elements.

We have $\frac{\partial^2 \hat{u}_i(m)}{(\partial y_i)^2} = \frac{\partial^2 v_i(y)}{(\partial y_i)^2} - 2 = r_i \frac{\partial^2 v_i(\hat{x}_i)}{(\partial \hat{x}_i)^2} - 2$ and since $v_i(\hat{x}_i)$ is strictly concave w.r.t. x_i, $\frac{\partial^2 v_i(\hat{x}_i)}{(\partial \hat{x}_i)^2} < 0$ and therefore, $\frac{\partial^2 \hat{u}_i(m)}{(\partial y_i)^2} < 0$ (note that r_i is not dependent to any of agent i's messages.).

We can also see that $\frac{\partial^2 \hat{u}_i(m)}{(\partial q_i^{j,l})^2} = -2$ and $\frac{\partial^2 \hat{u}_i(m)}{(\partial n_i^{j,l})^2} = -2$ and $\frac{\partial^2 \hat{u}_i(m)}{(\partial p_i^l)^2} = -2$ Since all of the diagonal elements of H are negative and non-diagonal elements are zero, matrix H is negative definite and hence $\hat{u}_i(y_i, n_i, q_i, p_i, m_{-i})$ is strictly concave w.r.t. m_i. ■

Lemma 3: At any NE, $\tilde{m} \in \tilde{\mathcal{M}}$ we have

$$\tilde{q}_i^{j,l} = \tilde{y}_j^l, \quad \forall j \in I_i, l \in \mathcal{L}. \tag{13}$$

Proof: According to Lemma 2, the utility functions are strictly concave w.r.t. the messages and so the best response function of agent i at each m_{-i} is unique and is determined by setting the gradient of utility function w.r.t. m_i to zero, if possible, and if it's not possible for each of the elements of m_i, the best response would be the upper boundary message for positive derivative and lower boundary message for negative derivative. Since the message spaces do not have upper boundaries, in order to have a best response, the derivative has to be negative and the best response would be the lower boundary message which in our message space it would be zero. The elements of gradient can all be set to zero except the derivative w.r.t. y_i that may not get to zero. Hence, if at NE $\tilde{y}_i > 0$ or equivalently $\hat{x}_i(\tilde{m}) > 0$, the gradient of $\hat{u}_i(m_i, m_{-i})$

is zero and otherwise, if $\tilde{y}_i = 0$ or equivalently $\hat{x}_i(\tilde{m}) = 0$, the derivative of $\hat{u}_i(m_i, m_{-i})$ w.r.t y_i is negative and other elements of gradient are zero. Hence, by setting the derivative of $\hat{u}_i(m_i, m_{-i})$ w.r.t. elements of m_i including $q_i^{j,l}$ to zero we have

$$\frac{\partial \hat{u}_i(m_i, m_{-i})}{\partial q_i^{j,l}} = 0 \Rightarrow 2(q_i^{j,l} - y_j^l) = 0 \Rightarrow \tilde{q}_i^{j,l} = \tilde{y}_j^l \quad \forall i \in \mathcal{N}, j \in I_i, l \in \mathcal{L}.$$

Since $\tilde{y}_j^l \geq 0$ the above equation can always hold and the lemma is proved. ■

As mentioned earlier, each message component $q_i^{j,l}$ can be used as a proxy for y_j^l to act like $\tilde{y}_j^l$ at NE and yet, it is not determined by agent i.

Lemma 4: At any NE of game $\mathfrak{G}$,

$$\tilde{n}_i^{j,l} = \tilde{y}_j^l + \sum_{k \in \mathcal{N}(j), k \neq i} \tilde{n}_j^{k,l}. \tag{14}$$

This implies that at any NE,

$$\tilde{n}_i^{j,l} = \sum_{k \in \mathcal{N}, n(i,j)=k} \tilde{y}_k^l. \tag{15}$$

Proof: Just like the proof of Lemma 3, we set the derivative of $\hat{u}_i(m_i, m_{-i})$ w.r.t. each $n_i^{j,l}$ to zero

$$\frac{\partial \hat{u}_i(m_i, m_{-i})}{\partial n_i^{j,l}} = 0 \Rightarrow 2(n_i^{j,l} - y_j^l - \sum_{k \in \mathcal{N}(j), k \neq i} n_j^{k,l}) = 0$$
$$\Rightarrow n_i^{j,l} = y_j^l + \sum_{k \in \mathcal{N}(j), k \neq i} n_j^{k,l} \quad \forall i \in \mathcal{N}, j \in \mathcal{N}(i), l \in \mathcal{L}$$

Using a similar argument as the one used in [14, p. 131] it can be shown that $n_i^{j,l} = \sum_{k \in \mathcal{N}, n(i,j)=k} y_k^l$. ■

Lemma 5: (Primal Feasibility) At any NE of game $\mathfrak{G}$, the allocation vector $\hat{x}(\tilde{m})$ is feasible.

Proof: According to Lemmas 3 and 4, at any NE we have

$$f_i^l = \sum_{j \in \mathcal{N}} \tilde{y}_j^l$$

and so at any NE we have

$$\sum_{i \in \mathcal{N}^l} \hat{x}_i(\tilde{m}) = \sum_{i \in \mathcal{N}^l} r_i \tilde{y}_i \leq \frac{c^l}{\sum_{j \in \mathcal{N}} \tilde{y}_j^l} \sum_{i \in \mathcal{N}^l} \tilde{y}_i$$
$$= \frac{c^l}{\sum_{j \in \mathcal{N}} \tilde{y}_j^l} \sum_{i \in \mathcal{N}} \tilde{y}_i^l = c^l \quad \forall l \in \mathcal{L},$$

which implies the allocation at NE is feasible. ■

Lemma 6: (Dual Feasibility) At any NE of game $\mathfrak{G}$, $\tilde{p}_i^l \geq 0$ and also $\tilde{p}_i^l = \tilde{p}^l, \forall i \in \mathcal{N}, l \in \mathcal{L}_i$.

Proof: It is obvious by the definition of the message spaces that $\tilde{p}_i^l \geq 0$. To show the next part, we first derive the following equation,

$$\tilde{p}_i^l = \bar{p}_{-i}^l \quad \forall l \in \mathcal{L}, i \in \mathcal{L}_i.$$

Suppose it is not correct,

$$\exists l \in \mathcal{L}, i \in \mathcal{L}_i : \quad \tilde{p}_i^l \neq \bar{p}_{-i}^l.$$

Then there exists an agent $j \in \mathcal{N}^l : p_j^l > \bar{p}_{-j}^l$ (as an example we could consider the agent j with the highest p_j^l over all of the agents and if we have multiple choices, at least one of them will work.) and we show that agent j has a profitable deviation to ${p'}_j^l = \bar{p}_{-j}^l = \tilde{p}_j^l - \epsilon$. We can write

$$\hat{u}_j(\tilde{y}_j, \tilde{n}_j, q_j, {p'}_j^l, \tilde{p}_j^{-l}, \tilde{m}_{-j}) - \hat{u}_j(\tilde{y}_j, \tilde{n}_j, \tilde{q}_j, \tilde{p}_j, \tilde{m}_{-j})$$
$$= (\epsilon)^2 + \epsilon \bar{p}_{-j}^l (c^l - r_j f_j^l)^2 = \epsilon(\underbrace{\epsilon}_{>0} + \underbrace{\bar{p}_{-j}^l (c^l - r_j f_j^l)^2}_{\geq 0}) > 0, \tag{16}$$

therefore, we must have $\tilde{p}_i^l = \bar{p}_{-i}^l$.

As a result of this equality and because of the connectivity of $\mathcal{G}$, it is obvious that $\tilde{p}_i^l = \tilde{p}_j^l, \ \forall i, j \in \mathcal{N}$ and we denote this common price by $\tilde{p}^l$. ■

Lemma 7: (Complementary Slackness) At any NE of game $\mathfrak{G}$,

$$\tilde{p}^l (c^l - \sum_{i \in \mathcal{N}^l} \hat{x}_i) = 0 \quad \forall l \in \mathcal{L} \tag{17}$$

Proof: By setting the derivative of the utility function w.r.t. p_i^l to zero we have

$$\frac{\partial \hat{u}_i(m_i, m_{-i})}{\partial p_i^l} = 0 \Rightarrow \underbrace{2(p_i^l - \bar{p}_{-i}^l)}_{=0,\text{Due to lemma 6}} + \bar{p}_{-i}^l (c^l - r_i f_i^l)^2 = 0$$
$$\Rightarrow \tilde{p}^l (c^l - r_i f_i^l)^2 = 0 \Rightarrow \tilde{p}^l (c^l - r_i \sum_{j \in \mathcal{N}} \tilde{y}_j^l) = 0$$
$$\Rightarrow \tilde{p}^l (c^l - \sum_{i \in \mathcal{N}^l} \hat{x}_i) = 0.$$

■

Lemma 8: (Stationarity) At any NE of game $\mathfrak{G}$, the following constraints are satisfied,

$$v_i'(\hat{x}_i) = \sum_{l \in \mathcal{L}_i} \tilde{p}^l \quad \text{if} \quad \hat{x}_i > 0 \tag{18}$$

$$v_i'(\hat{x}_i) \leq \sum_{l \in \mathcal{L}_i} \tilde{p}^l \quad \text{if} \quad \hat{x}_i = 0. \tag{19}$$

Proof: According to the explanations at the beginning of the proof of Lemma 3, if $\hat{x}_i > 0$ we have

$$\frac{\partial \hat{u}_i(m_i, m_{-i})}{\partial y_i} = 0 \Rightarrow (\frac{\partial \hat{u}_i(m_i, m_{-i})}{\partial x_i}) \frac{dx_i}{dy_i} = 0$$
$$\Rightarrow (v_i'(x_i) - \sum_{l \in \mathcal{L}} \tilde{p}^l) r_i = 0 \Rightarrow v_i'(x_i) = \sum_{l \in \mathcal{L}} \tilde{p}^l,$$

and if $\hat{x}_i = 0$,

$$\frac{\partial \hat{u}_i(m_i, m_{-i})}{\partial y_i} \leq 0 \Rightarrow (\frac{\partial \hat{u}_i(m_i, m_{-i})}{\partial x_i}) \frac{dx_i}{dy_i} \leq 0$$
$$\Rightarrow (v_i'(x_i) - \sum_{l \in \mathcal{L}} \tilde{p}^l) r_i \leq 0 \Rightarrow v_i'(x_i) \leq \sum_{l \in \mathcal{L}} \tilde{p}^l.$$

Note that at NE, $r_i > 0$ and this is due to the fact that $\hat{x} = 0$ can never happen at NE. ■

Lemma 9: (Individual Rationality and Weak Budget Balance) At any NE, $\tilde{m}$, of the game $\mathfrak{G}$, individual rationality is satisfied, i.e.

$$v_i(\hat{x}_i(\tilde{m})) - \hat{t}_i(\tilde{m}) \geq v_i(0) - 0 \quad \forall i \in \mathcal{N}. \tag{20}$$

Also the sum of taxes satisfies

$$\sum_{i \in \mathcal{N}} \hat{t}_i(\tilde{m}) \geq 0 \tag{21}$$

that is weak budget balance.

Proof: First consider the weak budget balance equation. At NE, we can write $\hat{t}_i(\tilde{m}) = \hat{x}_i(\tilde{m}) \sum_{l \in \mathcal{L}_i} \tilde{p}^l$ and hence $\sum_{i \in \mathcal{N}} \hat{t}_i(\tilde{m}) \geq 0$.

Next, consider the individual rationality part, it is obvious for $\hat{x}_i(\tilde{m}) = 0$. For $\hat{x}_i(\tilde{m}) > 0$, we define the function u_i as

$$u_i(x) = v_i(x) - x \sum_{l \in \mathcal{L}_i} \tilde{p}^l.$$

Since $u_i(x)$ is concave w.r.t. x and $u_i'(\hat{x}_i(\tilde{m})) = 0$ and $u_i'(y) > 0$ for $0 < y < \hat{x}_i(\tilde{m})$, we can conclude $u_i(y) > u_i(0)$ and since $u_i(0) = v_i(0)$ and $u_i(\hat{x}_i(\tilde{m})) = v_i(\hat{x}(\tilde{m})) - \hat{t}_i(\tilde{m})$, it is obvious that $v_i(\hat{x}(\tilde{m})) - \hat{t}_i(\tilde{m}) \geq v_i(0)$. ■

Lemma 10: There exists a NE for the game $\mathfrak{G}$.

Proof: This is proved by showing that any allocation equal to the solution of problem (1) is a NE of the game $\mathfrak{G}$.

First note that since the valuation functions are monotonically increasing, the more the allocations are the better it is for all of the agents. So the solution of problem (1) is always on the boundaries of feasible region.

In order to prove the existence of NE in the game $\mathfrak{G}$, we show that any candidate message m with allocation $\hat{x}(m) = x^*$ and n_i, q_i and p_i satisfying lemmas 4 , 3 and 6 and $p_i^l = p^l = \lambda^{l^*} \forall i \in \mathcal{N}^l$ is a NE. First we show that such messages exist. Obviously, we can set the messages n_i, q_i and p_i so that the mentioned circumstances are satisfied. The demand vector y could be any multiple of x^* so that the allocation $\hat{x}(m)$ would be $\hat{x}(m) = x^*$. This is because of the fact that each message y is projected to the $\hat{x}(m)$ that is on the boundary of feasible region and is a scaled version of y.

Now by considering such message we must show that this is a NE. If $\hat{x}_i(m) > 0$, we proved that the gradient of $\hat{u}_i$ w.r.t. m_i is zero for all i and so every agent is best responding to others' messages and it's a NE. If $\hat{x}_i(m) = 0$ the gradient of utility function $\hat{u}_i(m_i, m_{-i})$ w.r.t. all elements of m_i except y_i is zero for all i and so those messages are best responses to m_{-i}. Also, we know that $\frac{\partial \hat{u}_i}{\partial y_i}|_{y_i=0} < 0$ and therefore, agent i can not profit by increasing y_i from zero to some positive value and hence, this is a NE. ■

Proof of Theorem 1: By using all of the lemmas in this section, we can now prove Theorem 1. Consider any $\tilde{m} \in \tilde{\mathcal{M}}$, the allocation vector, $\hat{x}(\tilde{m})$ and the variables $\tilde{p}^l$ satisfy all of the KKT conditions if $\hat{x}(\tilde{m})$ are used as the primal variables and $\tilde{p} = \{\tilde{p}^1, ..., \tilde{p}^L\}$ are used as the dual variables. Therefore, $\hat{x}(\tilde{m}) = x^*$ for any $\tilde{m} \in \tilde{\mathcal{M}}$ and so, full implementation is satisfied too. Furthermore, Lemma 9 proves individual rationality and weak budget balance. ■

V. Relaxing The Assumptions on Message Network

As mentioned in Section III, for the sake of simplicity of exposition, we have assumed that for every link l, the subgraph consisting of agents $i \in \mathcal{N}^l$ is a connected graph (Assumption 1). In order to relax this assumption, we will present two solutions separately One is in the expense of larger message space and the other is in the expense of more complicated mechanism but less increment in the message space dimensions. Both of these solutions will be explained in this section and the resulting changes in the mechanism will be discussed.

A. First Solution

The first solution is to modify the definition of message element p_i as $p_i = (p_i^l, l \in \mathcal{L})$. This is extending the price message announced by each agent i to be for every link in the network instead of only links that agent i is using. Hence, the definition of $\bar{p}_{-i}^l$ in (11) is changed as follows

$$\bar{p}_{-i}^l = \frac{1}{N(i)} \sum_{j \in \mathcal{N}(i)} p_j^l. \tag{22}$$

This necessitates a modification in the tax function definition for the case of $l \notin \mathcal{L}_i$ as

$$\begin{aligned} \hat{t}_i^l(m) = &\sum_{j \in \mathcal{N}(i)} (n_i^{j,l} - y_j^l - \sum_{k \in \mathcal{N}(j), k \neq i} n_j^{k,l})^2 + \sum_{j \in I_i} (q_i^{j,l} - y_j^l)^2 \\ &+ (p_i^l - \bar{p}_{-i}^l)^2 + (p_i^l - \bar{p}_{-i}^l)\bar{p}_{-i}^l (c^l - r_i f_i^l)^2 \end{aligned} \tag{23}$$

By these changes, all of the results in Section IV are valid by exchanging every term including $p_i^l, l \in \mathcal{L}_i$ with $p_i^l, l \in \mathcal{L}$. Hence, the mechanism is quite similar to the original one except that in the message elements, we have L price messages instead of L_i ones for each agent i.

The basic idea behind this solution is as follows. The assumption of connectedness of the subgraph of agents using link l (Assumption 1) was required in order for all users in this subgraph to be able to come to a consensus on the common price $\tilde{p}^l$ for using the link l. By letting each agent announcing prices for all links (not just the ones she is involved) this assumption is not needed anymore as this information can propagate through the entire graph (which is assumed to be connected).

B. Second Solution

In this solution, there is no need for every agent to quote a price for every link that she is not using. For every link l, we consider a connected subgraph $\mathcal{G}_l = (\mathcal{N}_l, \mathcal{E}_l)$ including all agents $i \in \mathcal{N}^l$ plus the minimum number of agents that do not use link l and are required to make the subgraph connected. We call it link l's subgraph. The important property of this subgraph is that it should be connected so that it can play the role that the whole graph plays in the mechanism of Section V-A. Since the graph $\mathcal{G}$ is connected, the subgraph $\mathcal{G}_l$ exists. For each agent i the set of links $l \notin \mathcal{L}_i$ which $i \in \mathcal{N}_l$ are denoted by $\mathcal{L}^i$ with $|\mathcal{L}^i| = L^i$. Now, we redefine the message

component p_i as $p_i = (p_i^l, l \in \mathcal{L}_i \cup \mathcal{L}^i)$, that is each agent i quotes a price for any link she is using and also for the links which she is on their subgraph. We define the set $\mathcal{N}_l(i)$ as

$$\mathcal{N}_l(i) = \{i \in \mathcal{N}(i) \cap \mathcal{N}_l\} \tag{24}$$

Just like previous notations, we denote the size of this set by $|\mathcal{N}_l(i)| = N_l(i)$. Next, the definition of $\bar{p}_{-i}^l$ is changed to:

$$\bar{p}_{-i}^l = \frac{1}{N_l(i)} \sum_{j \in \mathcal{N}_l(i)} p_j^l \tag{25}$$

The tax function should be modified as follows. It is the same for $l \in \mathcal{L}_i$. For $l \notin \mathcal{L}_i \cup \mathcal{L}^i$, the tax function is the same as the $l \notin \mathcal{L}_i$ case for the original mechanism. For $l \in \mathcal{L}^i$, we define the tax function as

$$\begin{aligned}\hat{t}_i^l(m) = &\sum_{j \in \mathcal{N}(i)} (n_i^{j,l} - y_j^l - \sum_{k \in \mathcal{N}(j), k \neq i} n_j^{k,l})^2 + \sum_{j \in I_i} (q_i^{j,l} - y_j^l)^2 \\ &+ (p_i^l - \bar{p}_{-i}^l)^2 + (p_i^l - \bar{p}_{-i}^l)\bar{p}_{-i}^l (c^l - r_i f_i^l)^2\end{aligned} \tag{26}$$

This solution results in the same lemmas that we had in Section IV for the original mechanism. We should just exchange every term including $p_i^l, l \in \mathcal{L}_i$ with $p_i^l, l \in \mathcal{L}_i \cup \mathcal{L}^i$.

VI. Conclusion

In this paper, we consider the problem of rate allocation between strategic agents in unicast transmission. The important new feature of this setting is that messages are not broadcasted (as in the standard MD literature) but agents have to obey the message communication constraints imposed by an underlying message communication network. The dimensionality of the agent i's message in the first case (with simplifying assumption on the message network) is $M_i = 1 + N(i)L + |I_i|L + L_i$. Since the function $\Phi(i)$ chooses one agent j, the average size of the sets I_i would be 1. Hence, the average size of each agent's message is $1 + \mathbb{E}_{i \in \mathcal{N}}(N(i))L + L + \mathbb{E}_{i \in \mathcal{N}}(L_i)$ and by denoting $\mathbb{E}_{i \in \mathcal{N}}(N(i))$ and $\mathbb{E}_{i \in \mathcal{N}}(L_i)$ by $\bar{N}$ and $\bar{L}$ respectively, we can have

$$\mathbb{E} \sum_{i \in \mathcal{N}} M_i = N(1 + L(\bar{N} + 1) + \bar{L}). \tag{27}$$

Clearly, the dimensionality of message space grows linearly with N.

For the second mechanism, the length of message is $\mathbb{E} \sum_{i \in \mathcal{N}} M_i = N(1 + L(\bar{N} + 2))$ which is larger than the first case because of having L instead of $\bar{L}$. This is due to the price message expansion. For the third mechanism, the length of message is $\mathbb{E} \sum_{i \in \mathcal{N}} |m_i| = N(1 + L(\bar{N} + 1) + \bar{L} + \bar{\bar{L}})$ where we denote $\bar{\bar{L}} = \mathbb{E}_{i \in \mathcal{N}} L^i$.

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